Maps of toric varieties in Cox coordinates

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Abstract

The Cox ring provides a coordinate system on a toric variety analogous to the homogeneous coordinate ring of projective space. Rational maps between projective spaces are described using polynomials in the coordinate ring, and we generalise this to toric varieties, providing a unified description of arbitrary rational maps between toric varieties in terms of their Cox coordinates. Introducing formal roots of polynomials is necessary even in the simplest examples.

Contents

1 Introduction

This paper describes maps between toric varieties in terms of Cox coordinates, that is, using the usual generators of the Cox rings of the source and target (§1.2.1 summarises background on Cox rings). The results are not confined to maps that preserve the toric structures, but to arbitrary rational maps of such varieties.

Any rational map between two projective spaces can be lifted to a morphism between their affine GIT covering spaces, or their *Cox covers* as we will routinely say: it is described by a sequence of homogeneous polynomials of the same degree. To generalise this to maps between any toric varieties, we need descriptions which also use roots of polynomials, and so we cannot hope to lift the maps to morphisms, or even to rational maps, between covering spaces: instead, we consider multi-valued maps like $x \mapsto \pm\sqrt{x}$, which we denote by $x \mapsto\!\!\!\Lleftarrow \sqrt{x}$ to emphasise that it is not a map in the usual sense.

Our main result, stated more precisely as Theorem 1.1 and in final form as Theorem 4.20, is this. Let $\varphi\colon X \dashrightarrow Y$ be a rational map between toric varieties (not necessarily respecting their toric structures). Then there is a 'multi-valued map' $\Phi\colon \mathbb{C}^m \longrightarrow\!\!\!\Lleftarrow \mathbb{C}^n$ between the Cox covers of X and Y which is defined using radical expressions in the Cox coordinates of X and has the following properties:

Evaluation at points: if φ is defined at $x \in X$ and $\xi \in \mathbb{C}^m$ is an expression for $x = [\xi]$ in Cox coordinates, then $\varphi(x) = [\Phi(\xi)] \in Y$.

Pullback of divisors: If $D = (f)$ is a Cartier divisor on Y, where f lies in the Cox ring $S[Y]$ of Y, then the divisors $\varphi^*(D)$ and $(\Phi^* f)$ on X agree on the open subset where φ is regular.

These are the two essential properties of the complete description Φ, refined as properties A–F in Sections 4–5, but it has other good features: for example, it allows easy computation of the image and preimage of subschemes under φ.

In the rest of this introduction we work out some easy examples and briefly survey enough of the Cox ring approach to toric geometry to be able to state the main result more precisely. Section 2 explains a class of radical extensions of rings which we apply in Section 3 to make a basic theory of multi-valued maps.

These two sections are the technical heart of the paper. The practical theory for describing maps that we build on this is natural, but it succeeds because we work in carefully controlled extensions of the Cox rings when writing the coordinates of maps. In Section 4, we say what it means for a multi-valued map to describe a rational map between toric varieties, and we prove the main Theorem 4.20 on the existence of a complete description Φ. Section 5 explains the composition of descriptions and the computation of images and preimages. We conclude in Section 6 with some considerations of computation and more substantial examples.

In this paper, we work over the complex numbers $\mathbb{C}$. The foundational aspects of toric geometry [KKMSD73] work over any field (though often assumed to be algebraically closed), but our presentation relies on Cox's construction [Cox95b], and that is given over $\mathbb{C}$.

1.1 Motivating examples

1.1.1 A line on a quadric

Consider the weighted projective space $\mathbb{P}(1,1,2)$ with homogeneous coordinates y_1, y_2, y_3. The coordinate axis $\Gamma = (y_2) \subset \mathbb{P}(1,1,2)$ is a smooth rational curve $\Gamma \cong \mathbb{P}^1$. In coordinates x_1, x_2 on $\mathbb{P}^1$, we can describe the embedding $\mathbb{P}^1 \to \Gamma \subset \mathbb{P}(1,1,2)$ by

$$[x_1, x_2] \mapsto [\sqrt{x_1}, 0, x_2]$$

(see §3.1 for our formal definition of $\sqrt{x_1}$). Multiplying through by $\sqrt{x_1}$ with the given weights $(1,1,2)$ gives an alternative:

$$[x_1, x_2] \mapsto [x_1, 0, x_1x_2].$$

We discuss why we think the first one is better.

The first issue is to calculate images of points. For instance, to see the image of the point $[0,1] \in \mathbb{P}^1$ using the first description, we immediately compute $[0,0,1]$. With the second description, we are in trouble, because the description of the map evaluates to $[0,0,0]$ and so does not explain anything; we could go to an affine piece to find the answer. The square root is not too bad. The image of the point $[1,0] \in \mathbb{P}^1$ computed by the first description is either $[1,0,0]$ or $[-1,0,0]$ depending on which root we take; but these are the same point in $\mathbb{P}(1,1,2)$, so either expression is fine.

The second issue is to pull back divisors. For instance, to pull back a Cartier divisor from the linear system of $\mathcal{O}_{\mathbb{P}(1,1,2)}(2)$, we would like simply to substitute the definining equations of the map. For example, suppose we pull back $y_3 = 0$. Clearly, this coordinate axis meets Γ transversely in one point $[1,0,0]$. Using the first description, we pullback the function y_3 to get the function x_2, whose vanishing locus on $\mathbb{P}^1$ is exactly $[1,0]$ as we would like. The second description,

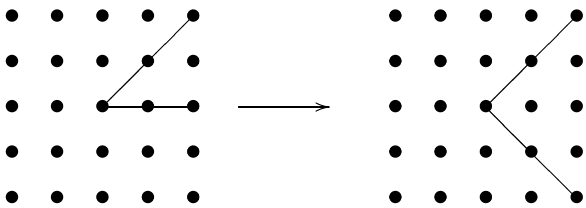
however, is not good enough in this respect either: the naive pull back is $x_1 x_2$. Again we might go to an affine piece to find the answer.

1.1.2 Weighted blow ups: the affine $\frac{1}{2}(1,1)$ singularitity

In the first example, the square root merely simplified some calculations. Now we give an example where it is unavoidable. Consider the simpliest singular toric variety Y: the affine $\frac{1}{2}(1,1)$ singularity, that is, the quotient of $\mathbb{C}^2$ by $\mathbb{Z}/2$ acting by

$$(y_1, y_2) \mapsto (-y_1, -y_2).$$

Let X be an affine piece of its resolution, $X = \mathbb{C}^2 \subset \mathrm{Bl}_{[0,0]} Y$. In fan terminology this corresponds to the following embedding of cones:



The map $\varphi \colon X \to Y$ as a map of affine varieties,

$$\varphi \colon \operatorname{Spec} \mathbb{C}[x_1, x_2] \to \operatorname{Spec} \mathbb{C}[{y_1}^2, y_1 y_2, {y_2}^2],$$

corresponds, via the dual map of cones, to the affine coordinate ring homomorphism:

$$\varphi^* \colon \mathbb{C}[{y_1}^2, y_1 y_2, {y_2}^2] \to \mathbb{C}[x_1, x_2]$$
$$\text{sending} \quad {y_1}^2 \mapsto x_1, \quad y_1 y_2 \mapsto x_1 x_2 \quad \text{and} \quad {y_2}^2 \mapsto x_1 {x_2}^2.$$

Therefore if we hope to extend φ^* to the full Cox ring $S[Y] = \mathbb{C}[y_1, y_2]$

$$\begin{tikzcd} \varphi^* \colon \mathbb{C}[{y_1}^2, y_1 y_2, {y_2}^2] \arrow[r] \arrow[d, hook] & \mathbb{C}[x_1, x_2] \arrow[d, hook] \\ \Phi^* \colon \mathbb{C}[y_1, y_2] \arrow[r] & \text{some new ring} \end{tikzcd}$$

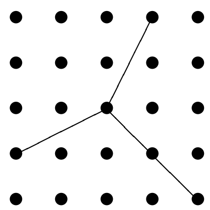
we need a map Φ^* doing either

$$\begin{array}{ll} y_1 & \mapsto \sqrt{x_1} \\ y_2 & \mapsto x_2 \sqrt{x_1} \end{array} \quad \text{or} \quad \begin{array}{ll} y_1 & \mapsto -\sqrt{x_1} \\ y_2 & \mapsto -x_2 \sqrt{x_1}. \end{array}$$

Introducing the square roots is necessary for such a description. We are allowed to choose either square root of x_1, but we must make the choice only once: having picked the root of x_1 for the first coordinate, the root of x_1 used in the second coordinate must be the same.

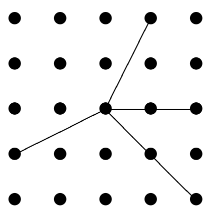
1.1.3 Fake weighted projective space

Descriptions of maps that require roots also arise for maps between projective toric varieties. Let Σ_Y be the fan



and Y the associated toric variety; this is the simplest example of a fake projective space (see [Buc08], [Kas09]) and is the quotient of $\mathbb{P}^2$ by $\mathbb{Z}/3$ acting with weights $(2, 1, 0)$.

Let X be a weighted blow up of any of the 3 singular points of Y, for example given by the fan



Then every description of the blow up map $X \to Y$ will involve at least 3rd roots of polynomials. For instance, if we encode the actions defining X and Y as

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & -3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

in coordinates $x_1, \ldots, x_4$ on X, $y_1, \ldots, y_3$ on Y—treating the second row of the weights of Y as the homogeneity imposed by the finite $\mathbb{Z}/3$ action—then the map is defined by

$$[x_1, \ldots, x_4] \mapsto [x_1 \sqrt[3]{x_4}^2, x_2 \sqrt[3]{x_4}, x_3].$$

(The second row of the grading matrix of Y only permits scaling by cube roots of unity, so it cannot be used to eliminate the radical here; the notation is slightly clumsy.) Global maps of this form are considered by Ahmadinezhad [Ahm10].

1.1.4 Ideals of subvarieties of toric varieties

The use of Cox rings to describe subschemes of toric varieties includes a small, well-known catch [Cox95b, Thm 3.7]: significantly different ideals can determine the same subscheme. This problem arises when considering maps too. Consider $X = \mathbb{P}^2$ and an action of $\mathbb{Z}/2$ on X with weights $(0, 0, 1)$. The quotient of X by $\mathbb{Z}/2$ is $Y = \mathbb{P}(1, 1, 2)$, and, in coordinates, the quotient map $\varphi\colon X \to Y$ is

$$[x_1, x_2, x_3] \mapsto [x_1, x_2, {x_3}^2].$$

This description of φ has the two properties mentioned at the outset (it is well defined on every point of X and Cartier divisors can be pulled back by simple substitution), but there is still a difficulty when calculating the preimage of subschemes in Y. For instance, in coordinates y_i on Y, the subschemes B_1 and B_2 of Y defined, respectively, by the following ideals

$$\langle y_1 \rangle \text{ and } \langle {y_1}^2, y_1 y_2 \rangle$$

are equal and both reduced, but the ideal defining B_2 is not radical even though both ideals are saturated at the irrelevant maximal ideal. Local calculations show that the preimage subscheme $A = \varphi^{-1}(B_2)$ is non-reduced and equal to scheme defined by $\langle {x_1}^2, x_1 x_2 \rangle$. On the other hand, if we pullback the defining equations of B_1 we get the reduced scheme $A' = (x_1)$. Although A and A' are certainly not equal as schemes, their scheme structures are equal on the preimage of smooth locus of Y. This is the best we can hope for and is explained generally in Theorem 5.9.

1.1.5 Using the descriptions of maps

Since our results apply to all rational maps, not only toric ones, we can apply them to compute spaces of maps, just as one does for usual maps to projective space. For example, consider the affine base of the standard 3-fold flop, namely Y with Cox ring $\mathbb{C}[y_1, \ldots, y_4]$, graded by $\mathbb{Z}$ with degrees $(1, 1, -1, -1)$ and empty irrelevant locus. We consider morphisms to Y from $\mathbb{P}^1$ with coordinates x_1, x_2. (Of course these will be trivial for this Y, but the point is the general method.) By Theorem 4.20, any such map can be written in coordinates as

$$\Phi\colon [x_1, x_2] \mapsto [f_1, \ldots, f_4]$$

for radical polynomial expressions f_i in x_1, x_2, that is, expressions of the form $g\sqrt[r]{h}$ for homogeneous polynomials g and h and some $r \in \mathbb{N}$; in particular, the degrees of the f_i are prima facie non-negative rational numbers. Assuming at first that the f_i are all nonzero, then since $y_1/y_2, y_1y_3, y_1y_4$ generate the function field $\mathbb{C}(Y)$, the homogeneity condition on Φ (which is what you would expect, but see Definition 4.9 for a full treatment) requires that each of $f_1/f_2, f_1f_3, f_1f_4$ is an honest rational function, irrespective of the radicals that may appear in the expressions for the f_i individually. In particular, f_1, f_3 and f_4 are all constants—since their degrees are all zero—from which it follows that f_2 is a constant too.

If any one of the f_i is zero, then the analysis is similar, but perhaps with a different choice of basis for $\mathbb{C}(Y)$. The same is true if $f_2 = f_4 = 0$; the pullback of y_1y_3 to f_1f_3 still forces these expressions to be constant. However, the case when $f_3 = f_4 = 0$ is slightly different. Homogeneity only forces f_1 and f_2 to have the same degree: instead, by Corollary 4.12, we may assume that they are both equal to zero, and so the map is the constant map to the origin in Y after all.

The variety Y is well known as the base of the standard flop: the two sides of the flop above Y are simply the two $\mathbb{Q}$-factorialisations of Y. In toric terms, these have the same Cox ring data as Y except the irrelevant ideal is not now trivial, but either

$$B = \langle y_1, y_2 \rangle \quad \text{or} \quad B = \langle y_3, y_4 \rangle .$$

We consider the first case. The analysis is the same as before, except when considering $f_3 = f_4 = 0$. The conclusion that all $f_i = 0$ is no longer valid, but instead $(x_1, x_2) \mapsto (f_1, f_2, 0, 0)$ can be any map from $\mathbb{P}^1$ onto a single image $\mathbb{P}^1$ in the target—the image curve is the flopping curve.

1.2 Maps of toric varieties in Cox coordinates

1.2.1 Cox coordinates on toric varieties

We review the standard elements of toric geometry that we use throughout this paper, closely following three of the standard sources [Cox95b], [Dan78] and [Ful93], without further comment or citation. A toric variety X of dimension d is defined by a fan Σ_X spanning a (possibly strict) subspace of a d-dimensional lattice N_X. The rays of Σ_X, which, by minor abuse of notation, we can take as the primitive points $\rho_1, \ldots, \rho_m$ on the 1-skeleton $\Sigma_X^{(1)}$, play two roles. First, treating them as independent symbols, they generate a new lattice $R_X \cong \mathbb{Z}^m$, **the ray lattice** of X, with chosen basis the ρ_i. There is a natural map $\rho_X \colon R_X \to N_X$ sending each symbol ρ_i to its embodiment. When considering the Cox quotient construction, one usually assumes for convenience that X has no torus factors, but this is not necessary in our approach (see also [CLS, §5.1]). If $X = X' \times (\mathbb{C}^*)^k$, where X' is has no torus factors, then the fan Σ_X spans a linear subspace $\langle \Sigma_X \rangle \subset N_X \otimes \mathbb{R}$ of codimension k. We choose primitive lattice vectors $\rho_{m+1}, \ldots, \rho_{m+k}$ in N_X such that the lattice $\langle \Sigma_X \rangle \cap N_X$ together with $\rho_{m+1}, \ldots, \rho_{m+k}$ generate the lattice N_X. These additional lattice vectors are called **virtual rays** and they play the role of place holders for variables corresponding to coordinates on $(\mathbb{C}^*)^k$. The ray lattice is then extended to $R_X \cong \mathbb{Z}^{m+k}$ with the bigger basis $\rho_1, \ldots, \rho_{m+k}$, and the map $\rho_X \colon R_X \to N_X$ is extended accordingly to take account of these virtual rays.

Second, we denote the elements of the basis dual to the ρ_i in R_X by x_i, and interprete them as the indeterminates of a polynomial ring. The ring the x_i generate is the famous **Cox ring** $S[X]$ of X, also known as its homogeneous, or total, coordinate ring. It is graded by the divisor class group $\mathrm{Cl}(X)$. The **irrelevant ideal** $B_X \subset S[X]$ is defined by standard generators, one for each maximal cone $\sigma \in \Sigma_X$, defined as $\mu_\sigma = \prod x_i$, where the product is taken over those rays ρ_i not contained in σ (one sets $B_X = S[X]$ if there is only one cone of maximal dimension). Note that if ρ_i is a virtual ray then the monomial μ_σ is divisible by x_i for every σ.

Thus $X = \mathbb{C} \times \mathbb{C}^*$, determined by a fan with a single ray in $N_X = \mathbb{Z}^2$ as its unique maximal cone, has Cox ring $S[X] = \mathbb{C}[x_1, x_2]$ and irrelevant ideal $B_X = \langle x_2 \rangle$ (rather than $S[X] = B_X = \mathbb{C}[x_1, x_2, 1/x_2]$, for example), where the variable x_1 corresponds to the 1-skeleton of the fan and x_2 corresponds to a virtual ray chosen arbitrarily to extend the rational span of the fan to the entire $\mathbb{Z}^2$.

We also treat the x_i in their own right, namely as a basis of the lattice dual to R_X, the **Cox monomials lattice** $TM(X)$. We write $TM[X]$ for the positive orthant in $TM(X)$. The lattice M_X of monomials, the dual of N_X, embeds $M_X \hookrightarrow TM(X)$ as the dual map to ρ_X.

The **Cox cover** of X is defined to be $\operatorname{Spec} S[X]$; it is isomorphic to $\mathbb{C}^m$ with standard coordinates x_i, and we usually write it as such with its heritage implicit. The gradings describe the action of a group $G_X = T \oplus A$, where $T \cong {\mathbf{G}_{\mathrm{m}}}^d$ is an algebraic torus and A is a finite abelian group. Cox proves, Theorem 2.1 of [Cox95b], that X is a quotient of $\mathbb{C}^m$ by G_X in the sense of GIT. Indeed, there is a rational map $\pi_X \colon \mathbb{C}^m \dashrightarrow X$ that is a morphism precisely on $\operatorname{Reg} \pi_X$, the complement of the **irrelevant locus** $\operatorname{Irrel}(X) = V(B_X) \subset \mathbb{C}^m$, and is a categorical quotient there. Thus one thinks of elements $\xi \in \mathbb{C}^m$ as representative coordinate expressions for their images $x = \pi_X(\xi) \in X$; we also denote $\pi_X(\xi)$ by $[\xi]$. These are the Cox coordinates on X that we use systematically.

Denoting the field of fractions of $S[X]$ by $S(X)$, the function field $\mathbb{C}(X)$ of X is naturally isomorphic to the subfield of $S(X)$ comprising functions that are invariant under G_X. We think of these as being the rational functions on $\mathbb{C}^m$ of degree 0, just as for rational functions on projective space, although the degree is really a character for G_X so naturally interpreted as a vector of integers and rationals modulo integers, and in any case the field $S(X)$ does not admit a direct sum decomposition into graded pieces by G_X. We refer to elements of $S[X]$ and $S(X)$ as **polynomial and rational sections** on X respectively, rather than as functions. We say that section $f \in S(X)$ is regular on $U \subset X$ if f is a regular function on $\pi_X^{-1}(U) = \{\xi \in \operatorname{Reg} \pi_X \mid \pi_X(\xi) \in U\}$.

The Cox ring has a more intrinsic definition. Suppose in the first place that X has no torus factors. Then

$$S[X] = \bigoplus H^0(X, D)$$

where the sum is taken over the Weil class group $\mathrm{Cl}(X)$, with D being a representative Weil divisor in the particular class (chosen systematically so that multiplication is defined automatically). The natural isomorphism between these two descriptions follows from the association of a Weil divisor D_ρ to each ray ρ: D_ρ is the irreducible divisor supported on the image of $\{x_\rho = 0\} \subset \mathbb{C}^m$ in X, where x_ρ is the Cox coordinate corresponding to ρ. In general, when $X = X' \times (\mathbb{C}^*)^k$ with virtual rays $\rho_{m+1}, \ldots, \rho_{m+k}$,

$$S[X][x_{m+1}^{-1}, \ldots, x_{m+k}^{-1}] = \bigoplus H^0(X, D)$$

where

$$S[X][x_{m+1}^{-1},\ldots,x_{m+k}^{-1}] = \mathbb{C}[x_1,\ldots,x_m,x_{m+1},x_{m+1}^{-1},\ldots,x_{m+k},x_{m+k}^{-1}].$$

We take these isomorphisms as implicit, so for each homogeneous rational section $f \in S(X)$ there is a Weil divisor, denoted (f). The converse is also true and follows from the same isomorphism: if D is a Weil divisor on a toric variety X, then $D = (f)$ for some non-zero homogeneous function $f \in S(X)$. Moreover, in the case X has no torus factors, D is effective if and only if $f \in S[X]$; if X does have torus factors, the criterion is instead that $f \in S[X][x_{m+1}^{-1},\ldots,x_{m+k}^{-1}]$. In any case, if D is effective, then there exists non-zero homogeneous section $f \in S[X]$ such that $D = (f)$. This association also obeys the natural calculus: $(fg) = (f) + (g)$.

Given $f \in S[X]$, as well as considering the divisor (f) on X, we will also consider the zero set of f in the Cox cover $\mathbb{C}^m$ of X. To avoid confusion we will always denote this affine zero set by $\{f = 0\} \subset \mathbb{C}^m$.

1.2.2 The main results

The elementary examples of §1.1 are part of a general theory. The first result is that any rational map between toric varieties has a description by radicals of Cox coordinates.

Theorem 1.1. *Let X and Y be toric varieties over $\mathbb{C}$ with Cox rings $S[X] = \mathbb{C}[x_1,\ldots,x_m]$ and $S[Y] = \mathbb{C}[y_1,\ldots,y_n]$ and corresponding Cox covers $\mathbb{C}^m$ and $\mathbb{C}^n$.*

If $\varphi\colon X \dashrightarrow Y$ is a rational map, then there are homogeneous rational sections $q_i \in S(X)$ and an expression

$$[x_1,\ldots,x_m] \overset{\Phi}{\mapsto} \left[\sqrt[r_1]{q_1},\ldots,\sqrt[r_n]{q_n}\right],$$

which satisfies the following properties:

(i) *If $\xi \in \mathbb{C}^m$ and φ is regular at $x = [\xi]$, then $y = [\Phi(\xi)]$ is a well-defined point of Y and $\varphi(x) = y$.*

(ii) *If $D = (f)$ is a Cartier divisor on Y whose suport does not contain the image of φ, where $f \in S(Y)$, then φ^*D and (Φ^*f) are equal as divisors on X when restricted to the regular locus of φ.*

(iii) *If $A \subset X$ is a closed subscheme defined by a saturated ideal $I_A \lhd S[X]$, then the image $\varphi(A) \subset Y$ is defined by the preimage under Φ^* of the span of I_A in some extension of $S[X]$.*

(iv) *If $B \subset Y$ is a closed subscheme defined by an ideal $I_B \lhd S[Y]$, then the preimage $\varphi^{-1}(B) \subset X$ is defined on $\varphi^{-1}(Y_0)$, by the ideal $\langle\Phi^*(I_B)\rangle \cap S[X]$ of $S[X]$, where Y_0 is the smooth locus of Y.*

This statement needs some explanation. In §4.1, we explain what it means for an expression

$$[x_1, \dots, x_m] \overset{\Phi}{\mapsto} [\sqrt[r_1]{q_1}, \dots, \sqrt[r_n]{q_n}],$$

to be a description of a rational map $X \dashrightarrow Y$, and Definition 4.18 specifies 'complete descriptions'. This theorem gathers some results for complete descriptions proved in Theorems 4.20, 5.9, 5.13, Proposition 5.1 and their subsequent comments and corollaries. Those results are more general and detailed; the statements above are special cases. The statement on preimage above does not explain the extension (in fact, it is simply a map ring $\Gamma(\Phi)$ as discussed next), but the precise details are in Corollary 5.14.

Furthermore, care is needed when defining $\Phi(\xi)$. Recall from §1.1.2 that the root of a polynomial can be chosen arbitrarily but only chosen once. If the same root of the same polynomial occurs again in the expression for Φ (even if not in an explicit form), then we must use the root chosen before. We make this book-keeping precise by introducing simple extensions of rings in §2.3 and map rings $\Gamma(\Phi)$ for Φ in §3.3. The point is that we work in extensions $\Gamma(\Phi)$ of $S[X]$ containing the image of Φ^* which cannot be made arbitrarily; the notion of 'simple' extension assembles just enough conditions for our purposes here. The ideal spans of the form $\langle J \rangle$ appearing in the statement are taken inside these $\Gamma(\Phi)$. Theorems 5.9 and 5.13 explain this precisely, and the latter also explains how to achieve the exact preimage over the nonsingular locus.

The second result gives a criterion for a radical expression like Φ above to determine a rational map of toric varieties; this is spelled out in Theorem 4.11.

Theorem 1.2. *Let $\Phi\colon \mathbb{C}^m \multimap \mathbb{C}^n$ be a multi-valued map between the Cox covers of toric varieties X and Y. If Φ satisfies the homogeneity and relevance conditions of Definition 4.9, then there is a unique rational map $\varphi\colon X \dashrightarrow Y$ that Φ describes.*

In other words, subject only to natural conditions of homogeneity with respect to all gradings and relevance (and the precise specification of what is allowed as a radical expression to define Φ), a sequence of radical expressions in Cox coordinates does indeed determine a rational map.

1.2.3 Other approaches to maps

The use of radical expressions to define maps is well established in some toric and orbifold contexts; computing weighted blow ups of (often terminal) quotient singularities is a common example. The radicals allow one to define a map on the orbifold cover, and this paper generalises such calculations systematically to all rational maps of toric varieties.

The Cox ring literature has several treatments of the functors both of toric varieties, initiated by Cox [Cox95a] and generalised by Kajiwara [Kaj98] (with

explicit use of radical expressions in examples), and of more general varieties satisfying certain finiteness conditions by Berchtold and Hausen [BH03]. These typically include additional hypotheses on the underlying varieties, and are concerned more with morphisms than with rational maps.

As well as considering rational maps in general, the treatment here contrasts with those above by considering all Weil divisors, not (sufficiently many) Cartier divisors. The key point is to build an extension of the Cox ring which contains the radicals used to describe a map. This 'map ring' provides an intermediate variety similar to a correspondence, and image and preimage calculations, for example, can proceed through that. Getting the right extension is delicate, but applying it is as natural as working with rational maps between projective spaces.

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2 Simple extensions of rings

We review some material in the context of multi-graded rings in §2.1, then present some field theory in §2.2, and finally give the key definition of simple extension of rings in §2.3.

2.1 Auxilliary algebra and geometry

2.1.1 Homogeneous ideals

We outline some standard points about ideals in graded rings with general finitely-generated abelian grading (semi-)groups. The cases we have in mind include the Cox ring $S[X]$ of a toric variety X, extensions $S[X][f^{-1}]$ for a homogeneous polynomial f, quotients $S[X]/I$ for some homogeneous ideal I and combinations of these. Recall that $S[X]$ has a distinguished ideal, the irrelevant ideal B_X. In our applications, the grading group $H = \mathrm{Hom}(G_X, \mathbb{C}^*)$ is invariably finitely-generated abelian. We write H additively and denote its identity by $0 \in H$; this is often used to say that an element g of one of the rings above has degree 0.

Definition 2.1. Let S be a graded ring. A homogeneous ideal $\mathfrak{p} \lhd S$ is **homogeneously prime** if and only if whenever a homogeneous $h \in \mathfrak{p}$ factorises $h = fg$ with homogeneous factors $f, g \in S$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

This notion is also called G-prime; see [Per07], Remark 3.20, for instance, or [Lud86] in an unrelated context. A homogeneous ideal which is prime is homogeneously prime. The converse is not always true.

Example 2.2. Let X be the affine $\frac{1}{2}(1,1)$ singularity, so $S[X] = \mathbb{C}[x_1, x_2]$ graded by $\mathbb{Z}/2$ as multiplication by -1 and $B_X = S[X]$. The ideal generated by $x_1^2 - 1$ is homogeneously prime but not prime. Indeed although it determines an irreducible line $L \subset X$, when regarded on the GIT cover $\mathbb{C}^2$ it determines a disjoint union of two lines, $x_1 = 1$ and $x_1 = -1$, which make up the preimage ${\pi_X}^{-1} L$.

This is an important consideration throughout the theory. However, when the grading group is $\mathbb{Z}^k$ with no torsion, the two concepts coincide.

Proposition 2.3. *Let S be a graded domain with grading group $\mathbb{Z}^k$. If $\mathfrak{p} \lhd S$ is a homogeneously prime ideal of S, then $\mathfrak{p}$ is a prime ideal.*

Proof. Assume $f, g \in S$ with fg homogeneous. To show that f and g are homogeneous, choose an order on the grading group of S and let f_1, g_1 be the lowest degree terms of f and g, and f_2, g_2 be the highest degree terms. If f or g is not homogeneous, then $f_1 g_1$ and $f_2 g_2$ have different degrees.

$\square$

Proposition 2.4. *[Cox95b, Proposition 2.4] For homogeneously prime ideal $I \lhd S[X]$ there exists a unique irreducible subvariety $V(I) \subset X$, such that a section $f \in S[X]$ vanish identically on $V(I)$ if and only if $f \in I$.*

Conversely, for every irreducible subvariety $V \subset X$, there exists a homogeneously prime ideal $I(V) \lhd S[X]$ contained in the irrelevant ideal B such that $V(I(V)) = V$.

$\square$

Definition 2.5. Let S be a graded ring and $I \lhd S$ an ideal. The **homogenisation of** I is the biggest homogeneous ideal I^{hgs} contained in I.

It follows that I^{hgs} is the ideal generated by all the homogeneous elements in I. The following easy proposition contains the essential observation that an image of an irreducible variety is irreducible. We use this later to prove that certain multi-valued maps descend to honest regular maps between toric varieties, even though on the Cox rings the pathologies of Example 2.2 can occur.

Proposition 2.6. *Let S be a graded domain. If $\mathfrak{p} \lhd S$ is a prime ideal, then $\mathfrak{p}^{\mathrm{hgs}}$ is homogeneously prime. In particular, if R any domain and $\alpha\colon S \to R$ is any ring homomorphism, then $(\ker \alpha)^{\mathrm{hgs}}$ is homogeneously prime.*

If R is a ring and $I \subset R$ is any subset, then we use $\langle I \rangle$ or $\langle I \rangle_R$ to denote the ideal generated by set I. We use this notation very often in the case that $S \subset R$ is a subring and $I \lhd S$ is an ideal. Then $\langle I \rangle_R \lhd R$ is the extension of the ideal I in the ring R.

Definition 2.7. [Har77, I.3] Let S be a graded ring and let $\mathfrak{p} \lhd S$ be a homogeneously prime ideal. Then the set A of all homogeneous elements in S which are not in $\mathfrak{p}$ is multiplicative, and the **(homogeneous) localisation** $S_{(\mathfrak{p})}$ is defined to be the set of degree 0 elements in $A^{-1}S$. It is a local ring with maximal ideal $(\mathfrak{p} \cdot A^{-1}S) \cap S_{(\mathfrak{p})}$.

If $f \in S$ is homogeneous, define the **(homogeneous) localisation** $S_{(f)}$ to be the set of degree 0 elements in $S[f^{-1}]$. If $I \lhd S$ is a homogeneous ideal, then $I_{(f)}$ is the set of degree 0 elements in $\langle I \rangle_{S[f^{-1}]}$; equivalently,

$$I_{(f)} = \langle I \rangle_{S[f^{-1}]} \cap S_{(f)}.$$

When $S = S[X]$ is the Cox ring of a toric variety X and $Z \subset X$ an irreducible subvariety defined by a homogeneously prime ideal $I(Z) \lhd S[X]$, the localisation $S[X]_{(I(Z))}$ is equal to the local ring of point Z in the scheme X:

$$S[X]_{(I(Z))} = \{q \in \mathbb{C}(X) \mid Z \cap \operatorname{Reg} q \neq \emptyset\}.$$

This is analogous to the usual statement for Proj of an $\mathbb{N}$-graded ring: see [Har77, Prop. II.2.5(a)], for example. Localisation at an element f is also analogous to the case of usual Proj. Roughly, $S_{(f)}$ consists of global rational functions that are regular an open subset $X_f = X \setminus (f)$, but there are caveats. First, if X has nontrivial $\mathbb{C}^*$-factors, then we assume that the zero locus of f contains the resulting divisorial components of the irrelevant locus $\operatorname{Irrel}(X)$. Second, the open subset X_f is not necessarily affine, so regular functions on X_f might be scarce (or even all constant).

If $I \subset S$ is an ideal, then it is easy to compare ideal membership for $g \in S_{(f)}$:

$$g \in I_{(f)} \iff g = f^{-k} g_I \text{ for some homogeneous } g_I \in I \text{ and } k \in N.$$

Lemma 2.8. *The homogeneous localisation of ideals is additive: if $I, J \lhd S$ are two homogeneous ideals, then*

$$(I + J)_{(f)} = (I)_{(f)} + (J)_{(f)}.$$

Proof. If $g \in S_{(f)}$, then

$$\begin{aligned} g \in (I+J)_{(f)} &\iff g = f^{-k}(g_I + g_J) && \text{for some } g_I \in I_{k \deg f}, \\ &&& g_J \in J_{k \deg f} \text{ and } k \in \mathbb{N}; \\ &\iff g = f^{-k} g_I + f^{-l} g_J && \text{for some } g_I \in I_{k \deg f}, \\ &&& g_J \in J_{l \deg f} \text{ and } k, l \in \mathbb{N}; \\ &\iff g \in I_{(f)} + J_{(f)}. \end{aligned}$$

$\square$

Definition 2.9. An ideal $I \lhd S[X]$ is **relevant** if it does not contain any power of the irrelevant ideal B_X.

Note that if I is relevant, then I^{hgs} is relevant too.

Lemma 2.10. *Let X be a toric variety and $\mathfrak{p} \lhd S[X]$ a homogeneously prime ideal. Set $R = S[X]_{(\mathfrak{p})}$. If $\mathfrak{p}$ is relevant, then R and R^{-1} generate $\mathbb{C}(X)$.*

Proof. Let A be the set of all homogeneous elements in $S[X]$ which are not in $\mathfrak{p}$, so $R = (A^{-1}S[X])^0$. We consider the subset $\mu \subset A$ of monomials not in $\mathfrak{p}$; we will find enough elements to generate $\mathbb{C}(X)$ from that. We treat μ naturally as a subset $\mu \subset TM[X]$. In fact, since $\mathfrak{p}$ is homogeneously prime, μ forms a lattice cone in $TM(X)$ which is a face of the positive cone $TM[X]$.

Set μ^* to be the face of the positive cone of the ray lattice R_X that is dual to μ (that is, the span of the basis elements ρ_i for which the corresponding Cox variable x_i is not in μ). Let $(\mu^*)^\vee \subset TM(X)$ be the cone dual to μ^*, which is precisely

$$(\mu^*)^\vee = \{z - y \mid z \in TM[X], y \in \mu\}$$

so the localisation $A^{-1}S[X]$ contains all the monomials in $(\mu^*)^\vee$. Restricting only to those monomials of degree 0 with respect to the gradings is the same as taking the pullback via the principal divisor map $M_X \hookrightarrow TM(X)$, so to prove the claim it is enough to prove that this pullback is a cone of maximal dimension in M_X.

The pullback above is simply the dual of the image of μ^* in N_X under the ray lattice map. Since $\mathfrak{p}$ is relevant, this image cone is one of the cones in the fan, so it is strictly convex and therefore its dual is of maximal dimension, as required.

$\square$

Lemma 2.11. *Let $U \subset X$ be an open affine subset. Then there exists a homogeneous $h \in S[X]$ such that $U = X \setminus \operatorname{Supp}(h)$ and in this case $U = \operatorname{Spec} S[X]_{(h)}$.*

Proof. Since U is affine,

$$\begin{aligned}
U &= \operatorname{Spec}\{q \in \mathbb{C}(X) \mid q \text{ is regular on } U\} \\
&= \operatorname{Spec}\left\{\frac{f}{g} \mid f, g \in S[X]_d, \text{ for some } d \in \operatorname{Cl}(X), \text{ s.t. } g \text{ is invertible on } U\right\} \\
&= \operatorname{Spec}\left(A^{-1}S[X] \cap \mathbb{C}(X)\right) \text{ where } A = \{g \in S[X] \mid g \text{ is invertible on } U\} \\
&= X \setminus \left(\bigcup_{g \in A} \operatorname{Supp}(g)\right)
\end{aligned}$$

and so the complement of U is a closed subset of codimension 1, and so it is equal to $\operatorname{Supp}(h)$ for some h.

$\square$

2.1.2 Equations defining subschemes

It is worth clarifying the ways in which a subscheme can be defined by an ideal in the Cox ring, since the differences arise when considering images and preimages of subschemes.

Definition 2.12. Let X be a toric variety with Cox ring $S[X]$. If $I \lhd S[X]$ is a homogeneous ideal, then we write $R = S[X]/I$ for the graded quotient ring, and when $h \in S[X]$ we denote the element $h + I \in R$ by $\tilde{h}$.

Suppose $A \subset X$ is a closed subscheme.

- We say I **defines** A if for every affine open subset $X_h = X \setminus (h)$ for some homogeneous h in $S[X]$ we have equality of schemes:

$$A \cap X_h = \operatorname{Spec} R_{(\tilde{h})}.$$

- We say I **maximally defines** A if I defines A and $I' \subset I$ for any other $I' \lhd S[X]$ which defines A.

- We say I **freely defines** A if I defines A and I is generated by $f_1, \dots, f_k$ for some homogeneous $f_i \in S[X]$, such that f_i defines a Cartier divisor.

An immediate conclusion from the additivity of localisation of Lemma 2.8 is the additivity of defining ideals.

Lemma 2.13. *Suppose $A_1, A_2 \subset X$ are two closed subschemes defined by homogeneous ideals I_{A_1}, I_{A_2}, respectively. Then the scheme theoretic intersection $A_1 \cap A_2$ is defined by $I_{A_1} + I_{A_2}$.*

□

Example 2.14. Let $X = \mathbb{P}(1,1,2)$ with Cox homogeneous coordinates x_1, x_2, x_3 and let A be the coordinate locus $x_2 = 0$. Then the ideal $I_{\max} = \langle x_2 \rangle$ maximally defines A and $I_{\text{free}} = \langle x_1 x_2, {x_2}^2 \rangle$ freely defines A.

In practice, ideals maximally defining a subscheme are often the simpler ones, and they describe global properties of the scheme in question, whereas ideals freely defining a subscheme say more about local properties. For instance in the example above we immediately see that A is not a local complete intersection.

We also say that an ideal $I \lhd S[X]$ is **saturated** if $(I : B_X) = I$, or equivalently if the scheme in $\mathbb{C}^m$ defined by I has no components (not even embedded components) with support on $\operatorname{Irrel}(X) \subset \mathbb{C}^m$.

Proposition 2.15. *Let X be a toric variety and $A \subset X$ a closed subscheme. Then there exists a unique homogeneous ideal $I_{\max} \lhd S[X]$ maximally defining A. Moreover this ideal is saturated.*

If, furthermore, X has enough Cartier divisors (in the sense of [Kaj98]), then there exists a saturated ideal I_{free} freely defining A.

□

Recall that if X is $\mathbb{Q}$-factorial or X is quasiprojective, then it has enough Cartier divisors; in contrast, we work on a complete variety with no effective Cartier divisors at all in §6.2.4.

The saturatedness property is essential for accurate calculations of image of a subvariety.

Example 2.16. Let $X = \mathbb{P}^1 \times \mathbb{C}$, $Y = \mathbb{C}$ and let $\varphi\colon X \to Y$ be the projection described in coordinates as $\Phi(x_1, x_2, x_3) = (x_3)$. Then $S[X] = \mathbb{C}[x_1, x_2, x_3]$ with $B_X = \langle x_1, x_2 \rangle$. Let $I_{A_1} = \langle x_1 x_3, x_2 x_3 \rangle$ and $I_{A_2} = \langle x_3 \rangle$. Then I_{A_1} is not saturated, its saturation is I_{A_2} and the scheme theoretic image $\overline{\varphi}(A)$ of the scheme $A \subset X$ given by either of these ideals is equal to the scheme given by $\langle y_1 \rangle \lhd S[Y]$. This ideal is obtained as $(\varphi^*)^{-1}(I_{A_2})$ whereas $(\varphi^*)^{-1}(I_{A_1}) = \langle 0 \rangle$.

2.1.3 Rational maps

We review standard facts about the image of subschemes under rational maps. Let X and Y be two (irreducible) algebraic varieties with fields of rational functions $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. Suppose $A \subset X$ is a closed subscheme; we always denote the corresponding ideal sheaf by $\mathcal{I}_A \lhd \mathcal{O}_X$.

Given a rational map $\varphi\colon X \dashrightarrow Y$, we denote by $\operatorname{Reg}\varphi \subset X$ the maximal open subset on which φ is regular and by φ_{reg} the restricted (regular) map $\varphi|_{\operatorname{Reg}\varphi}$. Suppose $U \subset \operatorname{Reg}\varphi$ is an open subset. By definition, the scheme-theoretic image $\overline{\varphi}|_U(A) \subset Y$ of A under φ restricted to U is the minimal closed subscheme of Y such that $\varphi_{\mathrm{reg}}|_{A\cap U}$ factorises through $\overline{\varphi}|_U(A)$. Set theoretically, $\overline{\varphi}|_U(A)$ is supported on $\overline{\varphi_{\mathrm{reg}}(A \cap U)}$. We write $\overline{\varphi}(A)$ for $\overline{\varphi}|_{\operatorname{Reg}\varphi}(A)$.

For $Z \subset Y$ a closed irreducible subvariety let $\mathcal{O}_{Y,Z} \subset \mathbb{C}(Y)$ be the local ring of Z, that is, the subring of functions whose poles do not contain Z. This is a local ring with maximal ideal $\mathfrak{m}_{Y,Z}$ consists of functions which vanish on Z.

Proposition 2.17. *Let X and Y be algebraic varieties.*

(i) *Let $\varphi\colon X \dashrightarrow Y$ be a rational map and set $Z = \overline{\varphi}(X)$. Then Z is reduced and irreducible and φ determines a ring homomorphism $\varphi^*\colon \mathcal{O}_{Y,Z} \to \mathbb{C}(X)$ (defined by pulling back rational functions) with kernel $\mathfrak{m}_{Y,Z}$.*

(ii) *Conversely, suppose $R \subset \mathbb{C}(Y)$ is a subring such that R and R^{-1} generate $\mathbb{C}(Y)$ (as a ring). Then every ring homomorphism $\alpha\colon R \to \mathbb{C}(X)$ uniquely determines a rational map $\varphi\colon X \dashrightarrow Y$ such that $\varphi^*|_R = \alpha$ and $R \subset \mathcal{O}_{Y,Z}$, where $Z = \overline{\varphi}(X)$.*

(iii) *Let $\varphi\colon X \dashrightarrow Y$ be a rational map. If $A \subset X$ is a closed subscheme and $V \subset Y$ is an open affine subset, then*

$$\mathcal{I}_{\overline{\varphi}(A)}(V) = (\varphi^*)^{-1}\mathcal{I}_A(\varphi_{\mathrm{reg}}^{-1}V) \lhd \mathcal{O}_Y(V).$$

(iv) *With the setup as above, if $B \subset Y$ is a closed subscheme and $U \subset \operatorname{Reg}\varphi$ is an open affine subset, then*

$$\mathcal{I}_{\varphi_{\mathrm{reg}}^{-1}(B)}(U) = \langle \varphi^* \mathcal{I}_B \rangle \lhd \mathcal{O}_{\operatorname{Reg}\varphi}(U)$$

determines the ideal sheaf of the preimage of B, also denoted $\mathcal{I}_B \cdot \mathcal{O}_{\operatorname{Reg}\varphi}$ in this context.

These are standard; see [Har77, §I.4] and [EH00, §V.1.1], for example.

□

In general the calculation of ideal of image $\mathcal{I}_{\overline{\varphi}(A)}(V)$ in (iii) above will involve Gröbner type calculations, whereas the calculation of the preimage as in (iv) only involves evaluating polynomial functions. Yet in the very special case that A is a point of X, the Gröbner calculations may be avoided. In general terms, evaluating a map $\varphi\colon X \dashrightarrow Y$ at A is simple. Suppose X has some sort of coordinates $x_1, \ldots, x_m$ near A (homogeneous, affine, local or whatever else) with A corresponding to $[a_1, \ldots, a_m]$ in these coordinates, and Y has coordinates $y_1, \ldots, y_n$ at the image $\varphi(A)$. To find the coordinates of $\varphi(A)$ it is enough to evaluate the sequence $[\varphi^* y_1, \ldots, \varphi^* y_n]$ at $[a_1, \ldots, a_m]$. In other words, it makes sense to write that φ in coordinates is:

$$\varphi\colon [x_1, \ldots, x_m] \mapsto [\varphi^* y_1, \ldots, \varphi^* y_n].$$

More formally, if A corresponds to the maximal ideal sheaf $\mathfrak{m}_A$, then we have the evaluation map $ev_A\colon \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{m}_A \simeq \mathbb{C}$ and the image point $\varphi(A)$ corresponds to the kernel of the composition $ev_A \circ \varphi^*$ (if this kernel is a maximal ideal at all).

For descriptions of maps between toric varieties in Cox coordinates to be equally useful, we must be able to perform analogous operations. Although more involved than for maps of affine varieties or subvarieties in $\mathbb{P}^n$, essentially the same algorithms work for computing the image of the point and the preimage of a subscheme, and we carry out the proofs in §3.4.1 and §5.3.

The following proposition describes the locus where a rational map is regular in terms of regularity locus of the pullbacks of regular functions, and we use it later when proving the existence of 'complete' descriptions.

Proposition 2.18. *Let $\varphi\colon X \dashrightarrow Y$ be a rational map of irreducible varieties. Let $\{V_i\}$ be an affine cover of Y and I be the set of those i for which $V_i \cap \varphi(X)$ is nonempty. Let G_i be a set of generators of the affine coordinate ring $\mathcal{O}_{V_i}$. Then the locus where φ is regular is*

$$\operatorname{Reg}\varphi = \bigcup_{i \in I} \bigcap_{g \in G_i} \operatorname{Reg}\varphi^* g.$$

Proof. It is enough to assume that $Y = V_1$ is affine and then, by composing it with closed immersion into an affine space, that Y is an affine space and G_1 is the set of coordinate functions. In that case the statement is clear.

□

2.2 Field extensions

Throughout this subsection we assume $\mathbb{F}$ is a field which contains all the roots of unity (for example, $\mathbb{F}$ could contain an algebraically closed base field). We denote the algebraic closure of $\mathbb{F}$ by $\overline{\mathbb{F}}$. The main case we are interested in is $\mathbb{F} = \mathbb{C}(x_1, \ldots, x_m)$ or a finite extension of this. We refer to [Lan02, Chapter V] for standard material on field extensions.

Lemma 2.19. *Let $\gamma \in \overline{\mathbb{F}}$ be such that $\gamma^r \in \mathbb{F}$ for $r > 0$ and assume r is minimal with this property. Then the polynomial $t^r - \gamma^r \in \mathbb{F}[t]$ is the minimal polynomial of γ. In particular, the extension $\mathbb{F} \subset \mathbb{F}(\gamma)$ is of degree r.*

Proof. Let ϵ be a primitive r-th root of unity. Then in $\overline{\mathbb{F}}[t]$ we have

$$t^r - \gamma^r = (t-\gamma)(t-\epsilon\gamma)\cdots(t-\epsilon^{r-1}\gamma).$$

If $p \in \mathbb{F}[t]$ is the minimal polynomial of γ, then p divides $t^r - \gamma^r$ (see [Lan02, §V.1]). Hence (up to a scalar in $\mathbb{F}$) p must be a product of a subset of j of the factors of $t^r - \gamma^r$ above for some $0 < j \leq r$. But then $p(0) = \epsilon^N \gamma^j$ for some power N. Hence $\gamma^j \in \mathbb{F}$, and so by minimality of r we must have $j = r$ and $p = t^r - \gamma^r$ as claimed. The degree calculation follows by [Lan02, Prop. V.1.4].

□

Corollary 2.20. *Consider a sequence of field extensions*

$$\mathbb{F} = \mathbb{F}_0 \subset \mathbb{F}_1 \subset \cdots \subset \mathbb{F}_a = \mathbb{F}(\gamma_1, \ldots, \gamma_a)$$

where $\mathbb{F}_i = \mathbb{F}_{i-1}(\gamma_i)$ and each γ_i to some power is in $\mathbb{F}$. Set r_i to be the minimal positive integer such that ${\gamma_i}^{r_i} \in \mathbb{F}_{i-1}$. Then the collection

$$\left\{ {\gamma_1}^{j_1} \cdots {\gamma_a}^{j_a} \mid j_i \in \{0, \ldots, r_i - 1\},\ i \in \{1, \ldots, a\} \right\}$$

forms a basis of $\mathbb{F}(\gamma_1, \ldots, \gamma_a)$ as a $\mathbb{F}$-vector space.

Proof. Follows immediately from Lemma 2.19 and [Lan02, Prop. V.1.2].

□

Lemma 2.21. *Assume $\gamma_0, \ldots, \gamma_a \in \overline{\mathbb{F}}$ are all such that ${\gamma_i}^{r_i} \in \mathbb{F}$ for some $r_i > 0$ and $\gamma_0 + \cdots + \gamma_a = 0$. Then the set $\Xi = \{\gamma_0, \ldots, \gamma_a\}$ divides into a union of disjoint subsets*

$$\Xi = \Xi_1 \sqcup \cdots \sqcup \Xi_b$$

such that for each j all $\gamma \in \Xi_j$ are proportional over $\mathbb{F}$ and

$$\sum_{\gamma \in \Xi_j} \gamma = 0.$$

Proof. We argue by induction on a. If $a = 0$, then there is nothing to prove, so assume the result holds for all values less than $a \geq 1$ and that $\gamma_i \neq 0$ for every i.

Let r_i be the minimal positive integers for which $\gamma_i{}^{r_i} \in \mathbb{F}$ and let ϵ_i be a primitive r_i-th root of unity. Without loss of generality we may assume that r_0 is maximal among the r_i. By Lemma 2.19, $t^{r_i} - \gamma_i^{r_i} \in \mathbb{F}[t]$ is the minimal polynomial of γ_i.

Consider $\gamma_0 = -(\gamma_1 + \cdots + \gamma_a)$. The polynomial

$$q(t) = \prod_{\substack{j_1 \in \{0, \ldots r_1 - 1\} \\ \vdots \\ j_a \in \{0, \ldots r_a - 1\}}} (t + \epsilon_1{}^{j_1}\gamma_1 + \cdots + \epsilon_a{}^{j_a}\gamma_a)$$

is in $\mathbb{F}[t]$ and it vanishes at γ_0. Hence the irreducible polynomial $t^{r_0} - \gamma_0{}^{r_0}$ must divide $q(t)$. In particular

$$\epsilon_0(\gamma_1 + \ldots + \gamma_a) = \epsilon_1{}^{j_1}\gamma_1 + \cdots + \epsilon_a{}^{j_a}\gamma_a \quad \text{for some } j_1, \ldots, j_a.$$

Writing $\delta_i = (\epsilon_0 - \epsilon_i{}^{j_i})\gamma_i$ for $i \in \{1, \ldots a\}$, this equation becomes $\delta_1 + \cdots + \delta_a = 0$. Now, by the inductive assumption, the δ_i divide into groups $\Xi_1, \ldots, \Xi_b$, each of whose elements are proportional over $\mathbb{F}$, and for which

$$\sum_{\delta \in \Xi_k} \delta = 0 \quad \text{for each } k. \tag{2.22}$$

We consider three cases. In the first case, suppose there exist two different numbers i_1 and i_2, such that δ_{i_1} and δ_{i_2} belong to the same set Ξ_k and such that $\epsilon_0 - \epsilon_{i_1}{}^{j_{i_1}} \neq 0$ and $\epsilon_0 - \epsilon_{i_2}{}^{j_{i_2}} \neq 0$. Without loss of generality, assume $i_1 = a - 1$ and $i_2 = a$. Then

$$\begin{aligned} \gamma_{a-1} &= \frac{\delta_{a-1}}{\epsilon_0 - \epsilon_{a-1}{}^{j_{a-1}}} = g_{a-1}\delta \\ \text{and} \quad \gamma_a &= \frac{\delta_a}{\epsilon_0 - \epsilon_a{}^{j_a}} = g_a\delta \end{aligned}$$

for some $g_{a-1}, g_a \in \mathbb{F}$. So the a-tuple of elements

$$\gamma_0, \gamma_1, \ldots, \gamma_{a-2}, (\gamma_{a-1} + \gamma_a)$$

satisfies the conditions of the lemma and we use our inductive assumption to conclude. Note that γ_{a-1} and γ_a either form a new group on their own (if $\gamma_{a-1} = -\gamma_a$) or they both must be proportional to the elements of one of the groups existing by the inductive assumption.

In the second case, suppose that within a given group Ξ_k there is only one such i that $\epsilon_0 - \epsilon_i{}^{j_i} \neq 0$. Then from Equation (2.22) we deduce that $\gamma_i = 0$, contrary to our assumption.

Finally, as the third case, suppose that $\epsilon_0 = {\epsilon_i}^{j_i}$ for all i. In particular, r_0 divides r_i. But since we assumed r_0 to be maximal among r_i, we have

$$r_0 = r_i \quad \text{for all } i.$$

In summary, we have proved, under the inductive assumption, that for every $(a+1)$-tuple satisfying the hypotheses of the lemma, either all elements of the tuple are divided into the appropriate groups proportional over $\mathbb{F}$ or their minimal powers are all equal.

So consider the following $(a+1)$-tuple which also satisfies the hypotheses:

$$1, \frac{\gamma_1}{\gamma_0}, \ldots, \frac{\gamma_a}{\gamma_0}.$$

If it divides into the appropriate groups proportional over $\mathbb{F}$, then so do the γ_i. On the other hand if the minimal powers are equal, then all they are equal to 1 (because $1^1 \in \mathbb{F}$). In that case, $\frac{\gamma_i}{\gamma_0} \in \mathbb{F}$ and so again the γ_i are proportional over $\mathbb{F}$ (actually forming just one group Ξ_1 in this case).

$\square$

Corollary 2.23. *Let $k \subset \mathbb{F}$ be a subfield, and V be a k-vector space with a k-linear map $i\colon V \to \overline{\mathbb{F}}$ such that for every $\delta \in i(V)$ there is some $r > 0$ for which $\delta^r \in \mathbb{F}$. Then there exists $\gamma \in \overline{\mathbb{F}}$ and a k-linear map $j\colon V \to \mathbb{F}$, such that*

$$i(v) = j(v) \cdot \gamma \qquad \text{for all } v \in V.$$

We express the conclusion of this corollary by saying that the elements of $i(V)$ have a common irrational part.

Proof. If $\dim i(V) = 0$, then there is nothing to prove, so assume $\dim i(V) \geq 1$.

Fix a non-zero element $\gamma \in i(V)$ and take any other element $\delta \in i(v)$. Apply Lemma 2.21 for the triple $\gamma, \delta, -(\gamma + \delta)$ to conclude that

$$\delta = h(\delta) \cdot \gamma \qquad \text{for some } h(\delta) \in \mathbb{F}.$$

The implicit map h is clearly k-linear in δ, so define a k-linear map j by

$$j(v) = h(i(v)).$$

These j and γ have the required properties.

$\square$

2.3 Simple ring extensions

Let k be a field which contains all roots of unity and S an integral k-domain with field of fractions $\mathbb{F}$. Let $\overline{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$.

Definition 2.24. A ring Γ is called a **simple extension of** S if there exist $\gamma_1, \ldots, \gamma_a \in \overline{\mathbb{F}}$, with each ${\gamma_i}^{r_i} \in S$ for some $r_i > 0$ (which is assumed to be minimal), for which

(i) $\Gamma = S[\gamma_1, \ldots, \gamma_a]$,

(ii) Γ is a free S-module with basis $\left\{{\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a} \mid 0 \leq l_i < r_i\right\}$, and

(iii) For any δ in the field of fractions $K(\Gamma) \subset \overline{\mathbb{F}}$ of Γ, if $\delta^r \in S$ for some integer $r > 0$, then $\delta \in \Gamma$.

The elements $\gamma_1, \ldots, \gamma_a$ are called the **distinguished generators of Γ over S.**

As a corollary of the considerations in §2.2 we establish some basic properties of simple ring extensions. We state them in a slightly more general setting in preparation for Proposition 2.26 where we prove that our preferred class of extensions are simple extensions.

Corollary 2.25. *Let S, $\mathbb{F}$, Γ and the γ_i be as in Definition 2.24* (i) *and* (ii). *Let $K(\Gamma) \subset \overline{\mathbb{F}}$ be the field of fractions of Γ. Let $\delta \in K(\Gamma)$ be such, that $\delta^r \in \mathbb{F}$ for some r. Then:*

(i) *$K(\Gamma)$ is a vector space over $\mathbb{F}$ with basis $\left\{{\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a} \mid 0 \leq l_i < r_i\right\}$.*

(ii) *$\delta = g \cdot {\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a}$ for some $g \in \mathbb{F}$ and $0 \leq l_i < r_i$.*

(iii) *$\delta \in \Gamma$ if and only if $g \in S$, where δ is expressed in the basis as in* (ii). *In particular, $\Gamma \cap \mathbb{F} = S$.*

(iv) *Fix any $j \in \{1, \ldots, a\}$. Let Γ_{j-1} be the ring $S[\gamma_1, \ldots, \gamma_{j-1}]$ and let $K(\Gamma_{j-1})$ be its field of fractions. Then the polynomial $t^{r_j} - {\gamma_j}^{r_j}$ is irreducible in $K(\Gamma_{j-1})[t]$.*

Proof. To prove that (i) holds, observe that $K(\Gamma)$ is $\mathbb{F}$-generated by the listed elements because Γ is. On the other hand if there were an $\mathbb{F}$-linear relation between these generators, then after clearing the denominator there would be a relation between these S-generators of Γ, contradicting the assumption that Γ is the free module.

To prove (ii) using (i), write

$$\delta = \delta_1 + \cdots + \delta_b$$

where each δ_i is of the form $g_i \cdot {\gamma_1}^{l_{1,i}} \cdots {\gamma_a}^{l_{a,i}}$. Setting $\delta_0 = -\delta$ we can apply Lemma 2.21 to deduce that actually the δ_i divide into groups of elements proportional over $\mathbb{F}$ such that the sum in each group is 0. In particular, δ_0 must be either 0 or proportional over $\mathbb{F}$ to at least one of the δ_i, which finishes the proof of (ii). Part (iii) follows immediately from (ii).

In (iv), r_j is also the minimal positive integer, such that ${\gamma_j}^{r_j} \in K(\Gamma_{j-1})$, for otherwise, we would have an $\mathbb{F}$-linear relation between smaller powers of the γ_i contrary to (i). So the conclusion follows from Lemma 2.19.

$\square$

Our main concern is a particular class of simple extensions.

Proposition 2.26. *Let $S = \mathbb{C}[x_1, \ldots, x_m]$ be a polynomial ring. Let $g_1, \ldots, g_a$ be square free, pairwise coprime polynomials and $r_1, \ldots, r_a$ be positive integers. Set $\gamma_i = \sqrt[r_i]{g_i}$. Then $\Gamma = S[\gamma_1, \ldots, \gamma_a]$ is a simple extension of S with distinguished generators $\gamma_1, \ldots, \gamma_a$.*

Proof. Since the polynomials are pairwise coprime, there is no polynomial relation between the γ_j other than those generated by ${\gamma_j}^r - g_j = 0$. Thus Γ is a free module over S with the desired basis and Γ satisfies (i) and (ii) of Definition 2.24.

Suppose $\delta \in K(\Gamma)$ and $\delta^r \in S$ as in Definition 2.24(iii). Then by Corollary 2.25(ii) we can write $\delta = \frac{g}{h_1 \cdots h_b} \cdot {\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a}$, where g and the h_j are nonconstant polynomials in S, the h_j are irreducible, and none of the h_j divides g. We claim $b = 0$ so that the denominator does not exist, as required by Definition 2.24(iii). Suppose on contrary, that $b \geq 1$. Since the g_i are pairwise coprime, there is at most 1 of $g_1, \ldots, g_a$ which is divisible by h_1 (say this is g_i) and since g_i is square free, it can only divide h_1 with multiplicity 1. Thus the multiplicity of h_1 in $\delta^r \in S$ is $-r + \frac{r l_i}{r_i}$, which is always negative, a contradiction.

$\square$

We also observe that the simple extensions behave well under the localisation. (The proof is a simple verification of the definition, and we omit it.)

Proposition 2.27. *Suppose $S \subset \Gamma$ is a simple ring extension and that $f \in S$. Then $S[f^{-1}] \subset \Gamma[f^{-1}]$ is a simple ring extension with the same set of distinguished generators.*

$\square$

For $\delta \in K(\Gamma)$ with $\delta^r \in \mathbb{F}$ write $\delta = g \cdot {\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a}$ with $0 \leq l_i < r_i$ and $g \in \mathbb{F}$ as in Corollary 2.25(ii). Define **the floor** $\lfloor \delta \rfloor$ and **the ceiling** $\lceil \delta \rceil$ **of** δ to be

$$\lfloor \delta \rfloor := g \qquad \text{and} \qquad \lceil \delta \rceil := g \cdot {\gamma_1}^{\epsilon_1 r_1} \cdots {\gamma_a}^{\epsilon_a r_a}$$

respectively, where $\epsilon_i = \lceil l_i / r_i \rceil$ is either 0 (if $l_i = 0$) or 1 (if $l_i > 0$). They are both elements of $\mathbb{F}$, and are related by

$$\left\lfloor \frac{1}{\delta} \right\rfloor = \frac{1}{\lceil \delta \rceil}.$$

Proposition 2.28. *For δ as above, the floor and ceiling of δ satisfy both*

$$\delta \in \Gamma \iff \lfloor \delta \rfloor \in S, \qquad \textit{and} \qquad \delta \in \Gamma \Longrightarrow \lceil \delta \rceil \in S.$$

Moreover, if δ is an invertible element of Γ, then $\lfloor \delta \rfloor$ and $\lceil \delta \rceil$ are invertible elements of S.

□

So far we did not exploit the property (iii) of Definition 2.24. It is a normality condition, and it has two important consequences for us. First, if $\delta \in K(\Gamma)$ satisfies $\delta^r \in \mathbb{F}$ for some $r > 0$, then δ is regular on $\operatorname{Spec} S$ (as a multi-valued function, meaning that it has no poles; see Definition 3.3) if and only if $\delta \in \Gamma$. This is made precise in the proof of Lemma 5.2. Meanwhile we illustrate it with an example.

Example 2.29. Suppose $S := \mathbb{C}[x_1, x_2]$, and let $\Gamma' = S[\gamma]$ where $\gamma := \sqrt[4]{x_1 {x_2}^2}$. Then the extension $S \subset \Gamma'$ satisfies conditions (i)–(ii) of Definition 2.24, but does not satisfy (iii): for example, the multi-valued function $\delta := \sqrt{x_1} \in K(\Gamma')$ has no poles, but $\delta = \gamma^2/x_2 \notin \Gamma'$. Instead, we may consider a slightly bigger ring $\Gamma = S[\sqrt[4]{x_1}, \sqrt[2]{x_2}]$. Then $S \subset \Gamma$ is a simple extension and $\gamma, \delta \in \Gamma$.

The second consequence of 2.24(iii) is the uniqueness of $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ operations.

Proposition 2.30. *Suppose $\delta \in \overline{\mathbb{F}}$ is such that $\delta^r \in \mathbb{F}$. Then up to an invertible element in S, $\lfloor \delta \rfloor$ and $\lceil \delta \rceil$ are well defined elements of $\mathbb{F}$, independent on the choice of simple ring extension $S \subset \Gamma$ such that $K(\Gamma)$ contains δ.*

Proof. It is enough to prove the statement for $\lfloor \delta \rfloor$. More precisely, suppose $\Gamma := S[\gamma_1, \dots, \gamma_a]$ and $\Gamma' := S[\gamma'_1, \dots, \gamma'_b]$ are two simple ring extensions of S with $\delta \in \Gamma, \Gamma'$. Write $\delta = g \cdot {\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a} = g' \cdot {\gamma'_1}^{m_1} \cdots {\gamma'_b}^{m_b}$. We have to prove $g'/g \in S$ (inverting the roles of Γ and Γ' we also get $g/g' \in S$).

Observe that $\delta/g = {\gamma_1}^{l_1} \cdots {\gamma_a}^{l_a} \in \Gamma$, thus $(\delta/g)^r \in S$ for some r. By Definition 2.24(iii) also $\delta/g \in \Gamma'$. Since

$$\delta/g = (g'/g) \cdot {\gamma'_1}^{m_1} \cdots {\gamma'_b}^{m_b}$$

by Corollary 2.25(iii) we have $g'/g \in S$ as claimed.

□

The corollary below shows that for certain ideals in simple extensions $\Gamma \supset S$ the intersection with S is readily calculated. Exactly this kind of elimination is needed when using a description to compute the preimage of a scheme under a map of toric varieties, so it is important for computation to be able to sidestep the need for Gröbner type operations (see Proposition 5.11).

Corollary 2.31. *Let $I \lhd \Gamma$ be an ideal generated by $\delta_1, \ldots, \delta_\beta$ where each δ_i satisfies $\delta_i^{r_i} \in S$ for some $r_i > 0$. Then*

$$I \cap S = \left\langle \lceil \delta_1 \rceil, \ldots, \lceil \delta_\beta \rceil \right\rangle \lhd S.$$

In particular intersecting ideals generated by such δ_i in Γ with S is additive:

$$(I_1 + I_2) \cap S = (I_1 \cap S) + (I_2 \cap S).$$

Proof. This is repeated application of Lemma 2.32 below, keeping in mind Corollary 2.25(iv).

□

Lemma 2.32. *Let S be an integral domain. Consider an integral domain $\Gamma = S[\gamma]/\langle \gamma^r - g \rangle$ for some $r \in \mathbb{Z}$, $r > 0$ and $g \in S$ for which Γ is a free S-module with basis $1, \gamma, \ldots, \gamma^{r-1}$ (in particular, $\gamma^r - g$ is irreducible over S). Furthermore assume I is an ideal in Γ generated as*

$$I = \langle f_1, \ldots, f_\alpha, f_{\alpha+1}\gamma^{m_{\alpha+1}}, \ldots, f_\beta \gamma^{m_\beta} \rangle$$

where $f_i \in S$ and $0 < m_i < r$. Then

$$I \cap S = \langle f_1, \ldots, f_\alpha, f_{\alpha+1} g, \ldots, f_\beta g \rangle .$$

Proof. Clearly the listed generators are in $I \cap S$.

So consider $h \in I$:

$$h = \left(\sum_{i=1}^{\alpha} \sum_{j=0}^{r-1} h_{i,j} f_i \gamma^j \right) + \left(\sum_{i=\alpha+1}^{\beta} \sum_{j=0}^{r-1} h_{i,j} f_i \gamma^{j+m_i} \right)$$

for some $h_{i,j}$ in S. Rewrite h as:

$$h = \left(\sum_{i=1}^{\alpha} h_{i,0} f_i \right) + \left(\sum_{i=\alpha+1}^{\beta} h_{i,r-m_i} f_i g \right) + \gamma\left(\ldots\right) + \cdots + \gamma^{r-1}\left(\ldots\right).$$

If $h \in S$, then the summands with γ^i for $i \in \{1, \ldots, r-1\}$ are all 0 (because Γ is a free S module with basis $1, \gamma, \ldots, \gamma^{r-1}$). Hence:

$$h = \left(\sum_{i=1}^{\alpha} h_{i,0} f_i \right) + \left(\sum_{i=\alpha+1}^{\beta} h_{i,r-m_i} f_i g \right)$$

which is an element of

$$\langle f_1, \ldots, f_\alpha, f_{\alpha+1} g, \ldots, f_\beta g \rangle \lhd S$$

as claimed.

□

Lemma 2.33. *Let S, $\mathbb{F}$, Γ and the γ_i be as in Definition 2.24. Analogously, let Γ' be a simple extension of an integral k-domain S' and let $\mathbb{F}'$ be the field of fractions of S'. Assume $\Phi^*\colon S \to \Gamma'$ is a homomorphism. Then Φ^* can be extended (non-uniquely) to a homomorphism $\widetilde{\Phi^*}\colon \Gamma \to \overline{\mathbb{F}}$ as in the diagram:*

$$
\begin{tikzcd}
S \arrow[d, hook] \arrow[rrd, "\Phi^*"] & & S' \arrow[d, hook] \\
\Gamma \arrow[d, hook] \arrow[rrd, dotted, "\widetilde{\Phi^*}"] & & \Gamma' \arrow[d, hook] \\
\overline{\mathbb{F}} & & \overline{\mathbb{F}'}
\end{tikzcd}
$$

(so that, in particular, the diagonal square is commutative). The extension can be chosen as follows. For every i, suppose ${\gamma_i}^{r_i} \in S$ is the (minimal) defining property of γ_i and set $g_i := {\gamma_i}^{r_i}$. Then set

$$\widetilde{\Phi^*}(\gamma_i) := \sqrt[r_i]{\Phi^*(g_i)} \in \overline{\mathbb{F}'}$$

for any choice of the r_ith root.

Proof. Since the only polynomial relations between $\gamma_1, \ldots, \gamma_a$ are $g_i - {\gamma_i}^{r_i}$, $\widetilde{\Phi^*}$ really defines a homomorphism.

□

3 Roots and multi-valued maps

In this section we introduce the main technical tool to study descriptions of maps between toric varieties. We extend the field of rational functions to include special elements of its algebraic closure, so-called multi-valued functions. We use these multi-valued functions to define multi-valued maps in the same way rational functions are used to define rational maps.

We fix notation for this section, and indeed for the rest of this paper. We work with two toric varieties X and Y and their Cox covers:

$$
\begin{tikzcd}
\mathbb{C}^m = \operatorname{Spec} S[X] \arrow[d, dashed, "\pi_X"'] & \operatorname{Spec} S[Y] = \mathbb{C}^n \arrow[d, dashed, "\pi_Y"] \\
X & Y
\end{tikzcd}
$$

where $S[X] = \mathbb{C}[x_1, \ldots, x_m]$ and $S[Y] = \mathbb{C}[y_1, \ldots, y_n]$. Although in this section we work exclusively on the Cox covers $\mathbb{C}^m$ and $\mathbb{C}^n$, and everything could be described in the coordinates on these with no reference to X and Y, we maintain the connection between the Cox covers and their toric varieties in the notation.

3.1 Multi-valued sections

Definition 3.1. A **multi-valued section on** X is an element γ of the algebraic closure $\overline{S(X)}$. We say γ is **homogeneous** if $\gamma^r = f$ for some homogeneous $f \in S(X)$ and for some integer $r \geq 1$.

Notation 3.2. If γ is a homogeneous multi-valued section with $\gamma^r = f$ as above, then we write $\gamma = \sqrt[r]{f}$. It is implicit in this notation that r is minimal and that an r-th root of f has been chosen once and for all. Furthermore, other uses of $\sqrt[r]{f}$ in the same calculation will always be intended to refer to the same element γ.

The product and quotient of two homogeneous multi-valued sections is again homogeneous, but their sum is usually not: $\sqrt{x_1} + \sqrt{x_2}$ is not homogeneous even if x_1 and x_2 have the same degree. Furthermore, it is not true that every multi-valued section can be expressed as a sum of homogeneous ones.

In the first place, we treat multi-valued sections on X as mildly generalised rational functions on the affine Cox cover $\mathbb{C}^m$. In particular, we simply define when a homogeneous multi-valued section is regular or invertible on an open subset of $\mathbb{C}^m$ following the notions for rational functions.

Definition 3.3. Let $\gamma = \sqrt[r]{f}$ be a homogeneous multi-valued section of X with $f \in S(X)$ homogeneous. Then γ is **regular** if $f \in S[X]$. More generally, γ is **regular on** U, for a Zariski open subset $U \subset \mathbb{C}^m$, if f is regular on U. If γ is regular on U and does not vanish anywhere on U, we say γ is **invertible on** U.

The **domain of** γ, also called the **regular locus of** γ and denoted $\operatorname{Reg}\gamma$, is defined to be the largest open subset of $\mathbb{C}^m$ on which γ is regular.

If $V \subset X$ is a Zariski open subset of X and γ a homogeneous multi-valued section of X, then we say that γ is regular on V if it is regular on the open subset $\pi_X^{-1}(V) \subset \mathbb{C}^m$.

A typical homogeneous multi-valued section $\gamma = \sqrt[r]{f}$ is not a function in the usual sense. Nevertheless, for $\xi \in \operatorname{Reg}\gamma$ we write $\gamma(\xi)$ to denote the finite set of values $a \in \mathbb{C}$ for which $a^r = f(\xi)$.

Definition 3.4. A homogeneous multi-valued section $\gamma = \sqrt[r]{f}$ is **single valued** if $r = 1$, in which case $\gamma = f \in S(X)$.

This notion relies on the convention of 3.2 that r is assumed to be minimal. Thus, for example, $\sqrt[r]{1}$ is single valued, since the minimal choice $r = 1$ is possible. Since we are in characteristic 0 and our ground field contains all roots of unity, there is an equivalent set-theoretic condition; we omit the proof.

Proposition 3.5. *A homogeneous multi-valued section $\gamma \in \overline{S(X)}$ is single-valued if and only if $\gamma(\xi)$ has exactly one element for a general $\xi \in \operatorname{Reg}\gamma$.*

□

Finally we show that linear subspaces of homogeneous multi-valued sections all have the same irrational part. This is one of the key points that makes the theory work: if we imagined a map to projective space as being determined by a basis of a vector space of sections corresponding to a 'multi-valued linear system', then this property would allow us to divide out by the common irrational part to recover a map defined without radicals.

Proposition 3.6. *If V is a $\mathbb{C}$-vector space and $i\colon V \to \overline{S(X)}$ is a $\mathbb{C}$-linear map whose image consists of only homogeneous multi-valued sections, then there exists a homogeneous multi-valued section $\gamma \in \overline{S(X)}$ and a $\mathbb{C}$-linear map $j\colon V \to S(X)$ whose image consists of homogeneous elements of a constant degree, and*

$$i(v) = j(v) \cdot \gamma \quad \text{for all } v \in V.$$

Proof. If $\dim i(V) = 0$, then there is nothing to prove, so assume $\dim i(V) \geq 1$. Apply Corollary 2.23 for $k = \mathbb{C}$ and $\mathbb{F} = S(X)$ and let j' and γ' be the resulting map and section. Let $v_0 \in V$ be a vector such that $j'(v_0)$ is not zero and set $\gamma := \gamma' \cdot j'(v_0)$ and $j(v) := j'(v)/j'(v_0)$.

By assumption $(j(v)\cdot\gamma)^r$ is a homogeneous section in $S(X)$ for some r. Hence $\gamma^r \in S(X)$ and γ^r is homogeneous (take $v = v_0$). But then also $j(v)^r$ is homogeneous, being a quotient of two homogeneous sections, and so $j(v)$ is homogeneous too.

It remains to prove that $j(v_1)$ and $j(v_2)$ have the same degree for any $v_1, v_2 \in V$. Consider $j(v_1 + v_2) = j(v_1) + j(v_2)$. If $j(v_1)$ and $j(v_2)$ had different degrees, then the decomposition of $j(v_1 + v_2)$ into homogeneous components would have two components, but $j(v_1 + v_2)$ is also homogeneous, so there can be only one component.

□

3.2 Multi-valued maps

We define multi-valued "maps" between affine spaces allowing roots in their descriptions. We will not consider the largest possible class of maps that one might define by multi-valued sections, but only a particular case.

Definition 3.7. A **multi-valued map Φ from $\mathbb{C}^m$ to $\mathbb{C}^n$** is a $\mathbb{C}$-algebra homomorphism

$$\Phi^*\colon \mathbb{C}[\mathbb{C}^n] \to \overline{\mathbb{C}(\mathbb{C}^m)}$$

such that $\Phi^* y_i$ is a homogeneous multi-valued section for each $i = 1, \ldots, n$. We say Φ is **regular** on $U \subset \mathbb{C}^m$ if all $\Phi^* y_i$ are regular on U.

Notation 3.8. If Φ is a multi-valued map as above, then we write

$$\begin{aligned} \Phi\colon \mathbb{C}^m &\longrightarrow \mathbb{C}^n \\ \xi &\longmapsto \Big(\big(\Phi^* y_1\big)(\xi), \dots, \big(\Phi^* y_n\big)(\xi) \Big). \end{aligned}$$

Of course evaluating Φ at a point $\xi \in \mathbb{C}^m$ is slightly delicate. Each component is the evaluation of a multi-valued function, so it is a set. However $\Phi(\xi)$ is not necessarily the product of these sets, since we must match the roots appearing in the multi-valued sections when they are the same, as in §1.1.2. The evaluation will be explained in detail in §3.4.1.

We extend Φ^* to a subset of rational functions by the formula

$$\Phi^* \left(\frac{f}{g} \right) = \frac{\Phi^* f}{\Phi^* g} \qquad \text{whenever } \Phi^* g \neq 0.$$

If $q = f/g$ is a reduced expression and $\Phi^* g = 0$, then we say $\Phi^* q$ is not defined.

Example 3.9. The toric map of §1.1.2, an affine patch on the blow up of the affine quotient singularity $\frac{1}{2}(1,1)$, lifts to a multi-valued map

$$\begin{aligned} \Phi\colon \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (s,t) &\longmapsto (\sqrt{s}, \sqrt{s} \cdot t). \end{aligned}$$

Definition 3.10. Let φ be a rational map $\mathbb{C}^m \dashrightarrow \mathbb{C}^n$. We can naturally associate a multi-valued map Φ to φ, by assigning $\Phi^* := \varphi^*$. If a multi-valued map Φ arises in this way, then we say Φ is **single-valued**.

The maximal subset $U \subset \mathbb{C}^m$ on which Φ is regular is an open affine subset.

Proposition 3.11. *Let*

$$\begin{aligned} \Phi\colon \mathbb{C}^m &\longrightarrow \mathbb{C}^n \\ x &\longmapsto \left(\left(\frac{f_1}{g_1} \right)^{\frac{1}{r_1}}, \dots, \left(\frac{f_n}{g_n} \right)^{\frac{1}{r_n}} \right). \end{aligned}$$

be a multi-valued map. Assume that f_i/g_i is reduced for each i. Then the maximal subset $U \subset \mathbb{C}^m$ where Φ is regular is the complement of the vanishing locus of $g := g_1 \cdots g_n$: that is,

$$U = \big(\mathbb{C}^m\big)_g = \operatorname{Spec} S[X][g^{-1}].$$

In particular, since the g_i are homogeneous, U is G_X-invariant.

Proof. Clearly Φ is regular on $\left(\mathbb{C}^m\right)_g$. Further let ξ be such that $g_i(\xi) = 0$ for some i. Then Φ is not regular at ξ, because $\Phi^* y_i = \sqrt[r_i]{(f_i/g_i)}$ is not regular at ξ.

$\square$

Definition 3.12. Let $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be a multi-valued map. The **domain of** Φ, also called the **regular locus of** Φ and denoted $\operatorname{Reg}\Phi$, is defined to be the affine open subset U described in Proposition 3.11 above.

Corollary 3.13. *If a description Φ is determined by polynomial radicals*

$$x \longmapsto (\sqrt[r_1]{f_1}, \ldots, \sqrt[r_n]{f_n}),$$

for polynomials $f_1, \ldots, f_n \in S[X]$, then $\operatorname{Reg}\Phi = \mathbb{C}^m$.

3.3 Map rings of multi-valued maps

There is another natural way of thinking of a multi-valued map, and it is the key to the analysis here. Let $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be a multi-valued map. Recall the corresponding toric varieties X and Y. Choose $\Gamma(\Phi)$ to be any subring in $\overline{S(X)}$ which satisfies the following properties:

(i) $\Gamma(\Phi) = \mathbb{C}[\operatorname{Reg}\Phi][\gamma_1, \ldots, \gamma_a]$ for some homogeneous multi-valued sections $\gamma_1, \ldots, \gamma_a$, all of which are regular on $\operatorname{Reg}\Phi$.

(ii) the image $\Phi^*(S[Y]) = \Phi^*(\mathbb{C}[\mathbb{C}^n])$ is contained in $\Gamma(\Phi)$.

(iii) $S[X][\gamma_1, \ldots, \gamma_a]$ is a simple extension of $S[X]$ with distinguished generators $\gamma_1, \ldots, \gamma_a$ (so by Proposition 2.27 also $\Gamma(\Phi)$ is a simple extension of $\mathbb{C}[\operatorname{Reg}\Phi]$ with the same generators).

Although such rings are not uniquely determined, we give rings with these properties a name since they are so important in our considerations.

Definition 3.14. Any ring $\Gamma(\Phi)$ satisfying (i), (ii) and (iii) is called **a map ring of** Φ.

Proposition 3.15. *Let $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be a multi-valued map. Then there exists a map ring $\Gamma(\Phi)$ of Φ.*

This proof is constructive, but it does not necessarily give the most efficient way of choosing a map ring.

Proof. Let $\Phi^* y_i = \sqrt[r_i]{f_i}$, where $f_i \in S(X)$. Let $\{g_1, \ldots, g_a\}$ be a finite set of homogeneous, square free, and pairwise coprime, polynomials in $S[X]$ so that

each f_i has an expression as a Laurent monomial in the g_j. Set r to be lowest common multiple of all r_i. Then set

$$\gamma_j = \sqrt[r]{g_j} \quad \text{for all } j \in \{1, \dots, a\}\,.$$

We claim that $\Gamma(\Phi) = \mathbb{C}[\operatorname{Reg}\Phi][\gamma_1, \dots, \gamma_a]$ is a map ring of Φ.

Property (i) is satisfied by construction. It is also clear that each $\Phi^* y_i$ can be expressed in terms of the γ_j, so $\Gamma(\Phi)$ contains the image of $S[Y]$ which is Property (ii). Finally Property (iii) follows by Proposition 2.26 since the g_j are coprime.

□

Insisting that the map ring be a simple extension has three advantages which we describe in Section 5. First, the image of a point can be calculated by a simple evaluation. Second, it allows us to compose appropriate pairs of multi-valued maps. Finally, with the distinguished generators (which only need be calculated once for each map), preimages of subvarieties can be calculated, at least away from certain loci.

3.4 Images and preimages under multi-valued maps

Let $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be a multi-valued map and $\Gamma(\Phi)$ be a ring satisfying Conditions (i) and (ii) of §3.3. Eventually we need $\Gamma(\Phi)$ be a map ring of Φ, but for the sole purpose of proving Proposition 3.18 we consider this slightly more general object.

Setting $V(\Phi) = \operatorname{Spec}\Gamma(\Phi)$, we have two natural morphisms:

$$\mathbb{C}^m \xleftarrow{p_\Phi} V(\Phi) \xrightarrow{q_\Phi} \mathbb{C}^n,$$

where ${p_\Phi}^*$ is the inclusion of $S[X]$ in $\Gamma(\Phi)$ and ${q_\Phi}^*$ is defined by mapping y_i to $\Phi^* y_i \in \Gamma(\Phi)$. We treat $V(\Phi)$ informally as a correspondence (even though it is not constructed in the product, and in any case it is finite over $\operatorname{Reg}\Phi$ and not necessarily over $\mathbb{C}^m$). Using this, we can define (set-theoretic) image and preimage of subsets in a natural way.

Definition 3.16. Let $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ be a multi-valued map. Let $A \subset \operatorname{Reg}\Phi$ be a subset. The **image of A under Φ** is the subset of $\mathbb{C}^n$ defined by

$$\Phi(A) := q_\Phi\left({p_\Phi}^{-1}(A)\right).$$

Let $B \subset \mathbb{C}^n$ be a subset. The **preimage of B under Φ** is the subset of $\operatorname{Reg}\Phi$ defined by

$$\Phi^{-1}(B) := p_\Phi\left({q_\Phi}^{-1}(B)\right).$$

In Section 3.4.1 below, we explain how to evaluate a multi-valued function Φ at a point. This will be consistent with the notion of image just discussed in the case that $A = \{\xi\}$ consists of a single point: $\Phi(A) = \{\Phi(\xi)\}$. When Φ is a single-valued map, these definitions give the usual image and preimage under a rational map.

Since q_Φ is continuous and $p_\Phi \colon V_\Phi \to \operatorname{Reg}\Phi$ is finite and locally free (and thus closed by [Har77, Ex. II.3.5(b)] and open by [sta, Lemmas 042S and 02KB]), preimage behaves well with respect to the Zariski topology.

Proposition 3.17. *If $B \subset \mathbb{C}^n$ is open, then $\Phi^{-1}(B) \subset \mathbb{C}^m$ is open. If $B \subset \mathbb{C}^n$ is closed, then $\Phi^{-1}(B) \subset \operatorname{Reg}\Phi$ is closed.*

Proposition 3.18. *The definitions of image and preimage above are independent of the choice of $\Gamma(\Phi)$ satisfying conditions* (i) *and* (ii) *of §3.3.*

Proof. All the rings satisfying conditions (i) and (ii) must contain

$$\Gamma(\Phi)_{min} := \mathbb{C}[\operatorname{Reg}\Phi][\Phi^* y_1, \ldots, \Phi^* y_n].$$

On the other hand $\Gamma(\Phi)_{min}$ itself satisfies these two conditions. So for any $\Gamma(\Phi)$ we have the commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Spec}\Gamma(\Phi) \arrow[dl] \arrow[d] \arrow[dr] & \\
\operatorname{Reg}\Phi & \operatorname{Spec}\Gamma(\Phi)_{min} \arrow[l] \arrow[r] & \mathbb{C}^n
\end{tikzcd}$$

and since the middle vertical arrow is epimorphic it follows that it does not matter which way around one carries the subset between $\operatorname{Reg}\Phi$ and $\mathbb{C}^n$.

□

Later in §5.2–5.3 we will explain how to consider scheme-theoretic image and preimage under certain multi-valued maps. This is more delicate since the scheme structure of the image or preimage may depend on the choice of map ring $\Gamma(\Phi)$.

Proposition 3.19. *The ideal of the Zariski closure $\overline{\Phi(\operatorname{Reg}\Phi)}$ of $\Phi(\operatorname{Reg}\Phi)$ is the kernel of Φ^*.*

Proof. Since the image of p_Φ is exactly $\operatorname{Reg}\Phi$,

$$\Phi(\operatorname{Reg}\Phi) = q_\Phi\left({p_\Phi}^{-1}(\operatorname{Reg}\Phi)\right) = q_\Phi\big(V(\Phi)\big).$$

Now $f \in S[Y]$ vanishes on $q_\Phi\big(V(\Phi)\big)$ if and only if $0 = {q_\Phi}^* f = \Phi^* f$.

□

3.4.1 Image of a single point

We consider the image of a single closed point under a multi-valued map and prove that it can be computed by evaluation with a little care.

Example 3.20. Consider the following multi-valued map:

$$\begin{aligned} \Phi\colon \mathbb{C}^2 &\Rrightarrow \mathbb{C}^2 \\ (s,t) &\mapsto (\sqrt[6]{s}, \sqrt[2]{s^3}(t^2+s)). \end{aligned}$$

The image of the point $(64,-1)$ consists of the 6 points

$$(2\epsilon_6, 512\epsilon_6^3(1+64))$$

as ϵ_6 runs over the 6-th roots of unity. On the other hand, the point $(2, -512 \times 65)$ is not in the image of $(64,-1)$, even though $2 = \sqrt[6]{64}$ and $-512 \times 65 = -\sqrt{64^3}((-1)^2+64)$.

The crucial observation in this example is that the irrational parts $\sqrt[6]{s}$ and $\sqrt[2]{s^3}$ are algebraically dependent: in fact,

$$(\sqrt[6]{s})^4 = \sqrt[2]{s^3},$$

so $\sqrt[2]{s^3}$ is already in the extension ring $\mathbb{C}[s,t][\sqrt[6]{s}]$. (Some choice of the sixth root must have been made, and here we enforce that choice on the whole calculation.)

Choose a point $\xi \in \operatorname{Reg}\Phi$ and let $ev_\xi\colon \mathbb{C}[\operatorname{Reg}\Phi] \to \mathbb{C}$ be the evaluation map. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathbb{C}^n] \arrow[drr, "\Phi^*"] & & \mathbb{C}[\operatorname{Reg}\Phi] \arrow[r, "ev_\xi"] \arrow[d, hook] & \mathbb{C} \arrow[d, equal] \\
& & \Gamma(\Phi) \arrow[r, dotted, "\widetilde{ev}_\xi"] & \mathbb{C}
\end{tikzcd}$$

The extensions $\widetilde{ev}_\xi$ exist and they are precisely determined by any choice of roots of images of the distinguished generators (see Lemma 2.33).

Theorem 3.21. *Let $\xi \in \operatorname{Reg}\Phi$. Then $\Phi(\xi)$ is precisely the set of all those $\eta \in \mathbb{C}^n$ whose maximal ideal is a kernel of $\widetilde{ev}_\xi \circ \Phi^*$ for some extension $\widetilde{ev}_\xi$.*

Proof. Let $\mathfrak{m}_\xi = \ker ev_\xi \lhd \operatorname{Reg}\Phi$ be the maximal ideal of ξ. First assume $\eta \in \Phi(\xi)$. Then there exist a point $\zeta \in V(\Phi)$, such that $q_\Phi(\zeta) = \eta$ and $p_\Phi(\zeta) = \xi$. So if $\mathfrak{m}_\zeta \lhd \Gamma(\Phi)$ is the maximal ideal of ζ, then $\mathfrak{m}_\zeta \supset \langle \mathfrak{m}_\xi \rangle \lhd \Gamma(\Phi)$. Consider $ev_\zeta\colon \Gamma(\Phi) \to \Gamma(\Phi)/\mathfrak{m}_\zeta \simeq \mathbb{C}$. Now clearly $ev_\zeta|_{\mathbb{C}[\operatorname{Reg}\Phi]}$ is a (nonzero) ring homomorphism, whose kernel contains the maximal ideal $\mathfrak{m}_\xi$. So

$$ev_\zeta|_{\mathbb{C}[\operatorname{Reg}\Phi]} = ev_\xi,$$

and so $\widetilde{ev}_\xi := ev_\zeta$ is an extension of ev_ξ such that its kernel of $\widetilde{ev}_\xi \circ \Phi^*$ is $\mathfrak{m}_\eta$.

Now assume we have an extension $\widetilde{ev}_\xi$. Let $\mathfrak{m}_\zeta$ be its kernel. Clearly $\langle \mathfrak{m}_\xi \rangle \subset \mathfrak{m}_\zeta$, so $p_\Phi(\zeta) = \xi$ and therefore $q_\Phi(\zeta) \in \Phi(\xi)$.

□

4 Descriptions of maps

Consider two toric varieties X and Y and their Cox covers $\mathbb{C}^m = \operatorname{Spec} S[X]$ and $\mathbb{C}^n = \operatorname{Spec} S[Y]$, where $S[X] = \mathbb{C}[x_1, \ldots, x_m]$ and $S[Y] = \mathbb{C}[y_1, \ldots, y_n]$. In this section, we show how to use multi-valued maps $\Phi\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ to describe rational maps $\varphi\colon X \dashrightarrow Y$. In particular, we address

1. what it means for a multi-valued map Φ to describe a rational map φ.
2. which multi-valued maps describe rational maps at all.
3. that every rational map can be described by a multi-valued map.
4. a class of multi-valued maps that describe rational maps particularly well, or completely.

An algorithm for finding such a complete description Φ of a given φ is implicit in the proofs.

4.1 The agreement locus

Let $\Phi\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ be a multi-valued map. It fits into a diagram

$$\begin{tikzcd} \mathbb{C}^m \arrow[r, rightrightarrows, "\Phi"] \arrow[d, dashed, "\pi_X"] & \mathbb{C}^n \arrow[d, dashed, "\pi_Y"] \\ X & Y \end{tikzcd}$$

The regular locus $\operatorname{Reg} \Phi \subset \mathbb{C}^m$ of Φ, where its denominators do not vanish as in Definition 3.12, contains a finer subset, the locus where $\pi_Y \circ \Phi$ is a well-defined map of sets:

$$\operatorname{Reg}_Y \Phi := \{\xi \in \operatorname{Reg} \Phi \mid \Phi(\xi) \cap \operatorname{Reg} \pi_Y \neq \emptyset \text{ and } \#\pi_Y(\Phi(\xi)) = 1\}.$$

This locus $\operatorname{Reg}_Y \Phi$ may be empty. On the other hand, if $\operatorname{Reg}_Y \Phi$ contains a nonempty open subset, then we regard Φ as being adapted to Y; under this assumption, it makes sense to ask where Φ agrees with rational maps $X \dashrightarrow Y$.

Definition 4.1. Given a multi-valued map $\Phi\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ and a rational map $\varphi\colon X \dashrightarrow Y$, in the notation above, the **agreement locus of Φ and φ** is

$$\operatorname{Agr}(\Phi, \varphi) = \left\{\xi \in \operatorname{Reg}_Y \Phi \cap {\pi_X}^{-1}(\operatorname{Reg} \varphi) \mid \pi_Y \circ \Phi(\xi) = \varphi \circ \pi_X(\xi)\right\}.$$

In other words, the agreement locus is the set of points where both compositions $\pi_Y \circ \Phi$ and $\varphi \circ \pi_X$ are well-defined as maps of sets and they have the same values. The next definition is the key one.

Definition 4.2. We say Φ **is a description of** φ, or that Φ **describes** φ, if $\operatorname{Agr}(\Phi, \varphi)$ contains an open dense subset of $\mathbb{C}^m$.

When we have a multi-valued map Φ that describes a rational map φ, we say that φ is given in Cox coordinates by

$$\begin{aligned}\varphi\colon X &\dashrightarrow Y\\ x &\mapsto [(\Phi^* y_1)(x), \ldots, (\Phi^* y_n)(x)]\end{aligned}$$

without worrying that $\Phi^* y_i$ is really only evaluated on some $\xi \in \operatorname{Agr}(\Phi, \varphi)$ for which $x = [\xi]$.

Section 1.1 has several examples of descriptions of maps, and here is another.

Example 4.3. The diagonal embedding of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ has the description

$$\Phi\colon [x_1, x_2] \mapsto [x_1, x_2, x_1, x_2].$$

In this case, $\ker \Phi^* = \langle y_1 - y_3, y_2 - y_4 \rangle$ is not a homogeneous ideal with respect to the gradings

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ on the Cox coordinates } y_1, \ldots, y_4,$$

in contrast to the case of projective spaces. It is easy to see in this case that the homogeneous part of the kernel is $\langle y_1 y_4 - y_2 y_3 \rangle$, and that this defines the image of the embedding.

4.2 Homogeneity and relevance conditions

We consider when a multi-valued map describes some rational map. We prove an equivalence of four conditions that reflect the usual homogeneity conditions for maps between projective spaces. Together, they are referred to as *the homogeneity condition*.

Throughout this section we consider a multi-valued map Φ between the Cox covers of two toric varieties:

$$\begin{tikzcd} \mathbb{C}^m \arrow[r, "\Phi"] \arrow[d, dashed, "\pi_X"] & \mathbb{C}^n \arrow[d, dashed, "\pi_Y"] \\ X & Y \end{tikzcd} \tag{4.4}$$

Proposition 4.5. *Let Φ be a multi-valued map as in* (4.4) *above and*

$$T = \{y_i \mid i \in \{1, \ldots, n\} \text{ and } \Phi^* y_i \neq 0\}$$

be the set of Cox generators of $S[Y]$ that pull back nontrivially under Φ.

The following conditions are equivalent:

(A1) *If $q \in S(Y)$ is homogeneous and $\Phi^* q$ is defined, then $\Phi^* q$ is a homogeneous multi-valued section on X.*

(A2) *If $q \in \mathbb{C}(Y)$ and $\Phi^* q$ is defined, then $\Phi^* q \in \mathbb{C}(X)$.*

(A3) *There exist rational monomials $t_1, \ldots, t_k$ generating $\mathbb{C}(Y) \cap \mathbb{C}(T) = \mathbb{C}(T)^0$ as a field extension of $\mathbb{C}$ such that $\Phi^* t_i$ are homogeneous single-valued sections of degree 0.*

(A4) *For all $\xi, \xi' \in \operatorname{Reg} \Phi$ with $\xi' \in G_X \cdot \xi$, if $\eta \in \Phi(\xi)$ and $\eta' \in \Phi(\xi')$ then $\eta' \in G_Y \cdot \eta$.*

(A4′) *There exists an open dense subset $U \subset \operatorname{Reg} \Phi$, such that for all $\xi, \xi' \in U$ with $\xi' \in G_X \cdot \xi$, if $\eta \in \Phi(\xi)$ and $\eta' \in \Phi(\xi')$ then $\eta' \in G_Y \cdot \eta$.*

A2 is the usual treatment of rational maps $X \dashrightarrow Y$ as a map of function fields, taking care with the domain in case the rational map is not dominant. We use this statement to construct a rational map from a description (see Theorem 4.11), and it is also convenient in calculations (see §6.2.3, for instance). A3 is the same condition expressed for a finite number of generators, which is useful when deciding whether an expression determines a rational map; we also use it to construct a description of a rational map (see Theorem 4.13).

A1 is used in §5.1 to define the composition of descriptions and to prove Proposition 4.17, calculating the dimension of the complement of the agreement locus. It is not much help for deciding whether a given expression determines a rational map, as the example in §6.2.5 illustrates. A4 is the geometric condition that Φ maps G_X-orbits into G_Y-orbits. This is a closed condition, which is expressed as A4′. A4 and A4′ are used to give conditions for a multi-valued map to be a description of some rational map (see Proposition 4.10 and Theorem 4.11) and in the calculations of agreement locus in §4.4.

In the first place we prove only the equivalence of A1, A2 and A3. The missing equivalences with A4 and A4′ will be shown after Lemma 4.7 is proved.

Proof of equivalence of A1, A2 **and** A3. Suppose A1 holds for Φ. Let $V \subset S(Y)$ be the subspace of homogeneous sections of degree 0 for which the pullback by Φ is defined. Denote the restriction of Φ^* to V by $i \colon V \to \overline{S(X)}$. Since $i(1) = 1$ is rational and has degree 0, Proposition 3.6 implies that all elements of $i(V)$ are rational and of degree 0. Therefore A2 holds for Φ.

Suppose A2 holds. Since $\mathbb{C}(T)^0 \subset S(Y)^0$, any monomial generating set $t_1, \ldots, t_k$ of $\mathbb{C}(T)^0$ satisfies A3 for Φ.

Suppose A3 holds for Φ; we show that A1 holds. Let $q \in S(Y)$ be any homogeneous function. Write

$$q = \frac{\mu_1 + \cdots + \mu_\alpha}{\nu_1 + \cdots + \nu_\beta},$$

where the μ_i and ν_j are monomial terms in $S[Y]$ with $\deg \mu_i = d_1$ and $\deg \nu_j = d_2$ for all i and j. Assume that $\Phi^*(\nu_1 + \cdots + \nu_\beta) \neq 0$, so $\Phi^* q$ is defined.

Certainly each $\Phi^*\mu_i$ is a homogeneous multi-valued section. Therefore the Laurent monomial μ_{i_1}/μ_{i_2} is homogeneous of degree 0 and either $\Phi^*(\mu_{i_1}) = 0$ or $\Phi^*(\mu_{i_2}) = 0$ or $\Phi^*(\mu_{i_1}/\mu_{i_2})$ is a nonzero homogeneous degree 0 rational section in $\mathbb{C}(X)$. In particular, for every i,

$$\Phi^*(\mu_i) = f_i \cdot \gamma$$

where $\gamma \in \overline{S(X)}$ is a fixed homogeneous multi-valued section (independent of i) and $f_i \in \mathbb{C}(X)$. So

$$\Phi^*(\mu_1 + \cdots + \mu_\alpha) = (f_1 + \cdots + f_\alpha)\gamma.$$

Similarly, $\Phi^*(\nu_1 + \cdots + \nu_\beta) = (g_1 + \cdots + g_\beta)\delta \neq 0$, for some $\delta \in \overline{S(X)}$ and $g_j \in \mathbb{C}(X)$. So

$$\Phi^*(q) = h \cdot \varepsilon$$

where $\varepsilon = \gamma/\delta \in \overline{S(X)}$ is homogeneous and $h = (\sum f_i)/(\sum g_j) \in \mathbb{C}(X)$. So $\Phi^*(q)$ is homogeneous and A1 holds.

□

Lemma 4.6. *Let Y be an irreducible normal algebraic variety and let D be a Weil divisor on Y. Suppose $h^0(Y, \mathcal{O}(kD)) = 1$ for all $k \geq 0$. Then D is linearly equivalent to trivial divisor.*

Proof. Consider $P = \operatorname{Proj} \bigoplus_{k=0}^{\infty} H^0(Y, \mathcal{O}(kD))$ and the rational map $\theta \colon Y \dashrightarrow P$. The image P is a projective variety with Hilbert function of a point, and so P is a single point and θ is regular. Therefore $\mathcal{O}_Y(D)$ is isomorphic to $\theta^*\mathcal{O}_P(1) = \mathcal{O}_Y$.

□

It is well known that a regular invariant function on an affine variety with an action of reductive group can only distinguish between closed orbits of the action. However, invariant rational functions encode much more information and (at least in our setup) there are enough of them to distinguish any orbits. This is possible because as well as either taking values in the base field or having poles, rational functions can be undetermined at points: x_1/x_2 at the origin of $\mathbb{C}^2$, for example. As a consequence, the set of zeroes of a rational function is not necessarily closed and we are able to produce, for instance, an invariant rational map which vanishes on one orbit, has poles on another orbit and is undetermined on the intersection of the closures of these orbits.

Let $\widetilde{\mathbb{P}^1}$ be the set consisting of all complex numbers and two other points, which we call "∞" and "$\mathfrak{undef}$":

$$\widetilde{\mathbb{P}^1} = \mathbb{C} \cup \{\infty, \mathfrak{undef}\}.$$

Every rational section $q \in S(X)$ determines uniquely a map $\tilde{q}$ of sets:

$$\tilde{q}\colon \mathbb{C}^m \to \widetilde{\mathbb{P}^1}$$
$$\xi \mapsto \begin{cases} q(\xi) & \text{if } q(\xi) \text{ is well defined,} \\ \infty & \text{if } q \text{ has a pole at } \xi, \\ \mathfrak{undef} & \text{if } \xi \text{ is in the intersection of the closures} \\ & \qquad \text{of the loci of poles and zeroes of } q. \end{cases}$$

Note that if $q = f/g$ for some coprime polynomials $f, g \in S[X]$ and $\xi \in \mathbb{C}^m$ is such that $f(\xi) = g(\xi) = 0$, then $\tilde{q}(\xi) = \mathfrak{undef}$. One could regard this trick as viewing a rational function as a regular map to the quotient stack $\mathbb{C}^2/\mathbb{C}^*$ rather than as a rational map to $\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$. However our argument below is elementary and the use of stacks is not necessary.

For the purpose of the following lemma, if $q \in S(X)$ and $\xi, \xi' \in \mathbb{C}^m$, then we say q is **equal in strong sense** at ξ and ξ' if and only if $\tilde{q}(\xi) = \tilde{q}(\xi')$. We denote this strong equality by $q(\xi) =^s q(\xi')$.

Lemma 4.7. *Suppose X and Y are toric varieties with Cox covers $\mathbb{C}^m$ and $\mathbb{C}^n$. Let $U \subset \mathbb{C}^m$ be an open dense subset.*

(1) *Fix $q \in S(X)$. Then*

$$\Big(\forall \xi, \xi' \in U \text{ s.t. } \xi' \in G_X \cdot \xi \quad q(\xi) =^s q(\xi')\Big) \quad \Longleftrightarrow \quad q \in \mathbb{C}(X).$$

(2) *Fix two points $\eta, \eta' \in \mathbb{C}^n$. Then*

$$q(\eta) =^s q(\eta') \text{ for every } q \in \mathbb{C}(Y) \quad \Longleftrightarrow \quad \eta' \in G_Y \cdot \eta.$$

Proof. Fix q as in (1) and let $g \in G_X$. Then $q(\xi) =^s q(g \cdot \xi)$ for every $\xi \in U \cap g^{-1} \cdot U$ if and only if $q = q \circ g$, and (1) follows.

Now fix η and η' as in (2). First suppose $\eta' = g \cdot \eta$ for some $g \in G_Y$. Then for any $q \in \mathbb{C}(Y)$, we have $q(\eta') =^s q(g \cdot \eta) = q(\eta)$.

Thus we are left with the only non-trivial implication. Suppose η and η' are such that $q(\eta) =^s q(\eta')$ for every $q \in \mathbb{C}(Y)$. let $A = \overline{G_Y \cdot \eta}$ and $A' = \overline{G_Y \cdot \eta'}$. Let $I_A, I_{A'} \lhd S[Y]$ be the homogeneous ideals of $A, A' \subset \mathbb{C}^n$, respectively.

Fix $d \in \operatorname{Cl} Y$, the grading group of $S[Y]$. One of the following situations happens:

(a) either $(I_A)_{kd} \supsetneq (I_A \cap I_{A'})_{kd} \neq 0$ for some $k \in \mathbb{N}$,

(b) or $\dim S[Y]_{kd} \geq 2$ and $(I_A)_{kd} \supsetneq (I_A \cap I_{A'})_{kd} = 0$ for some $k \in \mathbb{N}$,

(c) or $\dim S[Y]_{kd} = 1$ and $(I_A)_{kd} \supsetneq (I_A \cap I_{A'})_{kd} = 0$ for all $k \in \mathbb{N}$,

(d) or $(I_A)_{kd} = (I_A \cap I_{A'})_{kd}$ for some $k \in \mathbb{N}$ and since both ideals are radical, the same holds for $k = 1$.

We are going to exclude all the possibilities except (d).

Suppose h is a common divisor of all polynomials in $(I_A)_{kd}$ and that h is homogeneous. Then $(I_A)_{kd}$ is isomorphic to $(I_A)_{kd-\deg h}$ and $(I_A \cap I_{A'})_{kd}$ is isomorphic to $(I_A \cap I_{A'})_{kd-\deg h}$. In particular, in cases (a)–(c)

$$(I_A)_{kd-\deg h} \supsetneq (I_A \cap I_{A'})_{kd-\deg h}.$$

In case (a) choose h to be the greatest common divisor of all polynomials in $(I_A)_{kd}$ and let $f \in (I_A \cap I_{A'})_{kd-\deg h}$ be any non-zero polynomial. Then choose $g \in (I_A)_{kd-\deg h} \setminus (I_A \cap I_{A'})_{kd-\deg h}$, such that f and g are coprime. Set $q = f/g \in \mathbb{C}(Y)$. In this case $\tilde{q}(\eta) = \mathfrak{undef}$ whereas $\tilde{q}(\eta') = 0$. Thus $q(\eta) \neq^s q(\eta')$ contrary to our assumption.

Now assume (b) holds. Choose h to be the greatest common divisor of all polynomials in $S[Y]_{kd}$. Thus

$$(I_A)_{kd-\deg h} \supsetneq (I_A \cap I_{A'})_{kd-\deg h} = 0$$

and $\dim S[Y]_{kd-\deg h} \geq 2$. Choose $f \in (I_A)_{kd-\deg h}$ and $g \in S[Y]_{kd-\deg h}$ which are non-zero and coprime. Set $q = f/g \in \mathbb{C}(Y)$. Then either $\tilde{q}(\eta) = 0$ (if $g \notin I_A$) or $\tilde{q}(\eta) = \mathfrak{undef}$ (if $g \in I_A$), whereas $\tilde{q}(\eta') \in \mathbb{C} \setminus \{0\}$. Thus $q(\eta) \neq^s q(\eta')$ contrary to our assumption.

Now suppose (c) holds. Then $(I_A)_{kd} = S[Y]_{kd}$ and by Lemma 4.6 we have $d = 0$ and I_A contains the constants of $S[Y]$. Thus $I_A = S[Y]$ which contradicts that A is non-empty.

So the only remaining possibility is that (d) holds. Since d is chosen arbitrary and the ideals I_A and $I_A \cap I_{A'}$ are homogeneous, it follows that $I_A = I_A \cap I_{A'}$, and thus $A = A \cup A'$. In the same way we can prove $A' = A \cup A'$ and therefore $A = A'$, which only can happen if $\eta' \in G_Y \cdot \eta$.

$\square$

Having proved the lemmas above we are in position to complete the proof of Proposition 4.5.

Proof of equivalence of A1–A2 **and** A4, A4′ **in Proposition 4.5.** Suppose both A1 and A2 hold. Let $\xi, \xi' \in \operatorname{Reg} \Phi$ with $\xi' \in G_X \cdot \xi$ and $\eta \in \Phi(\xi)$, $\eta' \in \Phi(\xi')$, and $q \in \mathbb{C}(Y)$ be any rational function. We claim $q(\eta) =^s q(\eta')$. To prove the claim first suppose q is such that $\Phi^* q$ is defined. Then

$$\begin{aligned} q(\eta) &\in q(\Phi(\xi)) = (\Phi^* q)(\xi) \quad \text{and} \\ q(\eta') &\in q(\Phi(\xi')) = (\Phi^* q)(\xi'). \end{aligned}$$

By A2, $\Phi^* q$ is a rational function (in particular it is single valued), and so $(\Phi^* q)(\xi) = (\Phi^* q)(\xi')$ by Lemma 4.7(1). Hence $q(\eta) =^s q(\eta')$ as claimed.

If Φ^*q is not defined, then we can write $q = f/g$ where $f, g \in S[Y]$ and they are coprime homogeneous polynomials and $g|_{\Phi(\operatorname{Reg}\Phi)} \equiv 0$. Thus $\tilde{q}(\eta), \tilde{q}(\eta') \in \{\infty, \mathfrak{undef}\}$. Suppose by contradiction that $q(\eta) \neq^s q(\eta')$, and without loss of generality $\tilde{q}(\eta) = \infty$ and $\tilde{q}(\eta') = \mathfrak{undef}$. Then $f(\eta) \neq 0$ and $f(\eta') = 0$. But $\Phi^* f$ is a homogeneous multi-valued section by A1 and $0 \neq f(\eta) \in f(\Phi(\xi)) = \Phi^* f(\xi)$ and analogously $0 = f(\eta') \in \Phi^* f(\xi')$. Suppose r is such that $h := (\Phi^* f)^r \in S(X)$. Then $h(\xi) \neq 0$ and $h(\xi') = 0$, but h is homogeneous, so $h(\xi)$ and $h(\xi')$ may only differ by an invertible scalar, a contradiction. Thus also in this situation $q(\eta) =^s q(\eta')$.

The claim and Lemma 4.7(2) imply that the points η and η' are in the same orbit of G_Y as required for A4.

If A4 holds, then clearly A4$'$ holds too.

Finally, suppose A4$'$ holds. Let $q \in \mathbb{C}(Y)$ be such that Φ^*q is defined. Suppose $\xi \in U$. The possible values taken by Φ^*q at ξ are simply those values taken by q at the points of the image set $\Phi(\xi)$. Setting $\xi = \xi'$ in A4$'$ shows that $\Phi(\xi)$ is contained in a single G_Y-orbit, and so Φ^*q is single valued by Lemma 4.7(2) and Proposition 3.5. Therefore $\Phi^*q \in S(X)$. In any case, for any ξ, ξ' as in A4$'$,

$$(\Phi^*q)(\xi) = q(\Phi(\xi)) = q(\Phi(\xi')) = (\Phi^*q)(\xi')$$

since q is constant on G_Y-orbits. So Lemma 4.7(1) shows that the degree of Φ^*q is zero, and condition A2 holds.

$\square$

Next we note the equivalence of another three conditions, again jointly referred to as *the relevance condition*.

Proposition 4.8. *Let Φ be a multi-valued map as in* (4.4) *above and*

$$R_0 = \left\{ \rho_i \in \Sigma_Y^{(1)} \mid i \in \{1, \dots, n\} \text{ and } \Phi^* y_i = 0 \right\}$$

to be rays of the fan Σ_Y of Y which correspond to Cox generators of $S[Y]$ that pull back trivially under Φ.

The following conditions are equivalent:

B1 *The image of Φ is not contained in the irrelevant locus of Y.*

B2 $\ker \Phi^*$ *does not contain the irrelevant ideal B_Y of Y, that is,* $\ker \Phi^*$ *is a relevant ideal.*

B3 *The rays of R_0 are all contained in a single cone of Σ_Y.*

Proof. The equivalence of the first two conditions is immediate (even acknowledging the multi-values of Φ). If σ is a maximal cone of Σ_Y containing all the rays of R_0, then the standard generator $m_\sigma \in B_Y$ determined by σ satisfies $\Phi^* m_\sigma \neq 0$.

Thus m_σ is not contained in $\ker \Phi^*$, and so neither is B_Y. Conversely, if there is no maximal cone containing all the rays of R_0, then every standard generator of B_Y contains at least one such ray. Therefore $B_Y \subset \ker \Phi^*$.

$\square$

Definition 4.9. Let Φ be a multi-valued map as in (4.4) above.

A We say that Φ satisfies the **homogeneity condition** if any of the equivalent conditions A1, A2, A3, A4, A4′ of Proposition 4.5 hold for Φ.

B We say that Φ satisfies the **relevance condition** if any of the equivalent conditions B1, B2, B3 of Proposition 4.8 hold for Φ.

Proposition 4.10. *If Φ is a description of a rational map $\varphi\colon X \dashrightarrow Y$, then Φ satisfies the homogeneity and relevance conditions of Definition 4.9.*

Proof. By Definition 4.2 of description, $\pi_Y \circ \Phi$ is defined on an open subset of $\mathbb{C}^m$, so $\Phi(x)$ cannot be contained in the irrelevant locus for those points. Therefore Φ satisfies the relevance condition B1.

Since Φ is a description the agreement locus $\operatorname{Agr}(\Phi, \varphi)$ contains an open dense subset of $\operatorname{Reg}\Phi$. The homogeneity condition A4′ is satisfied on this set.

$\square$

The converse is the main point: the homogeneity and relevance conditions guarantee that a multi-valued map is a description of a uniquely-determined rational map.

Theorem 4.11. *Let Φ be a multi-valued map as in* (4.4) *above that satisfies the homogeneity and relevance conditions of Definition 4.9.*

(i) *By its action on rational functions, Φ^* naturally determines a rational map $\varphi\colon X \dashrightarrow Y$.*

(ii) *Φ is a description of some map $\psi\colon X \dashrightarrow Y$ if and only if $\psi = \varphi$.*

Proof. To prove (i) first note that $\mathfrak{p} := (\ker \Phi^*)^{\mathrm{hgs}} \lhd S[Y]$ is homogeneously prime by Proposition 2.6, so that the following localisation makes sense:

$$R := S[Y]_{(\mathfrak{p})}.$$

We claim Φ^* naturally determines a ring homomorphism:

$$\mathbb{C}(Y) \supset R \xrightarrow{\Phi^*} \mathbb{C}(X).$$

This is because by definition

$$R = \left\{ \frac{f}{g} \mid f, g \in S[Y],\ g \notin \mathfrak{p} \text{ and } f,\, g \text{ are homog. of the same degree} \right\}.$$

Since $\mathfrak{p}$ is generated by all homogeneous sections in $\ker \Phi^*$, we can also replace the condition $g \notin \mathfrak{p}$ with $g \notin \ker \Phi^*$:

$$R = \left\{ \frac{f}{g} \mid f, g \in S[Y],\ g \notin \ker \Phi^* \text{ and } f,\, g \text{ are homog. of the same degree} \right\}.$$

In particular, if $\frac{f}{g} \in R$, then the pull-back by Φ is defined (because $\Phi^* g$ is not zero).

By the homogeneity condition A2,

$$\Phi^* \left(\frac{f}{g} \right) \in S[X]_0 \cong \mathbb{C}(X).$$

So we have a ring homomorphism $R \to \mathbb{C}(X)$ as claimed.

Note, that by Lemma 2.10 together with the relevance condition B2, R and R^{-1} together generate $\mathbb{C}(Y)$. Hence by Proposition 2.17(ii) the ring homomorphism

$$\Phi^* \colon R \to \mathbb{C}(X)$$

determines a rational map $\varphi\colon X \dashrightarrow Y$ which is characterised by its action on rational functions $q \in \mathbb{C}(Y)$ being $\varphi^*(q) = \Phi^*(q)$.

Next we have to prove that Φ describes φ. Consider the open subset

$$U = \{\xi \in \operatorname{Reg} \Phi \mid \xi \notin \operatorname{Irrel}(X), \Phi(\xi) \nsubseteq \operatorname{Irrel}(Y)\}$$

of $\operatorname{Reg} \Phi$; note that it contains a non-empty open subset of $\mathbb{C}^m$ by the relevance condition. Choose any $\xi \in U$. By the homogeneity condition A4, $\pi_Y(\Phi(\xi))$ is a single point y. We claim $y = \varphi(\pi_X(\xi))$, so that $\xi \in \operatorname{Agr}(\Phi, \varphi)$.

To prove the claim, we set $x = [\xi] = \pi_X(\xi)$ and evaluate rational functions $q \in \mathbb{C}(Y)$ at $\varphi(x)$ and at y:

$$q(\varphi(x)) = (\varphi^* q)(x) = (\Phi^* q)([\xi]) = q([\Phi(\xi)]) = q(y).$$

So no rational function on Y can distinguish between $\varphi(x)$ and y and therefore $y = \varphi(x)$. Hence $U \subset \operatorname{Agr}(\Phi, \varphi)$ and Φ describes φ.

Finally we note that if $\psi\colon X \dashrightarrow Y$ is another rational map which is also described by Φ, then for $\xi \in \operatorname{Agr}(\Phi, \psi)$ with $x = [\xi]$ and for a rational function $q \in K(Y)$ we have

$$(\psi^* q)(x) = q(\psi(x)) = q([\Phi(\xi)]) = (\Phi^* q)(\xi) = (\varphi^* q)(x).$$

Hence $\psi^* = \varphi^*$ and therefore $\psi = \varphi$.

$\square$

Corollary 4.12. *Let Φ be a description of a rational map $\varphi\colon X \dashrightarrow Y$.*

(i) *Let $\sigma \in \Sigma_Y$ be the smallest cone which contains all rays whose corresponding coordinate y_i is pulled back to 0 by Φ. Then the toric stratum corresponding to σ is the smallest stratum of Y that contains $\varphi(X)$.*

(ii) *The assignment*

$$\Psi^* y_i := \begin{cases} 0 & \text{if the } i\text{-th ray of } \Sigma_Y \text{ is in } \sigma, \\ \Phi^* y_i & \text{otherwise} \end{cases}$$

defines a multi-valued map Ψ, and Ψ also describes φ.

(iii) *If, furthermore, Y is $\mathbb{Q}$-factorial, then $\Phi^* y_i = 0$ if and only if $\varphi(X)$ is contained in the locus $y_i = 0$.*

Proof. When $\pi_Y \colon \mathbb{C}^n \dashrightarrow Y$ is a geometric quotient, $\eta \in \mathbb{C}^n$ is a semistable point and y_i is a Cox coordinate, then

$$y_i(\eta) = 0 \iff \pi_Y(\eta) \in \mathrm{Supp}(y_i),$$

where (y_i) is the divisor on Y corresponding to y_i. So if Y is $\mathbb{Q}$-factorial and $\xi \in \mathrm{Agr}(\Phi, \varphi)$, then

$$\begin{aligned} \varphi(\pi_X(\xi)) \in \mathrm{Supp}(y_i) &\iff \pi_Y(\Phi(\xi)) \in \mathrm{Supp}(y_i) \\ &\iff y_i(\Phi(\xi)) = 0 \\ &\iff (\Phi^* y_i)(\xi) = 0, \end{aligned}$$

which proves the final statement.

If the quotient π_Y is not geometric, then we have only

$$y_i(\eta) = 0 \implies \pi_Y(\eta) \in \mathrm{Supp}(y_i),$$

so that $\varphi(X)$ is contained in the intersection of the supports of divisors (y_i) for those y_i with $\Phi^* y_i = 0$. On Y, this intersection is the toric stratum corresponding to the cone σ. In particular, $\varphi(X) \subset \mathrm{Supp}(z)$ for every Cox coordinate z corresponding to a ray of σ, whether or not $\Phi^* z$ is zero. So for any $\xi \in \mathrm{Agr}(\Phi, \varphi)$ we have $\pi_Y(\Phi(\xi)) = \pi_Y(\Psi(\xi))$. Since Ψ and Φ therefore have the same agreement locus, Ψ is also a description of φ.

□

4.3 Existence of descriptions

The previous section shows that descriptions of rational maps are characterised by the homogeneity and relevance conditions. Now we show that every rational map does have a description.

Theorem 4.13. *Let $\varphi\colon X \dashrightarrow Y$ be a rational map of toric varieties. Then there exists a description $\Phi\colon \mathbb{C}^m \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\rightrightarrows \mathbb{C}^n$ of φ.*

Proof. We construct $\Phi^* y_1, \dots, \Phi^* y_n$ inductively. Set $\Phi^* y_i = 0$ if and only if $\varphi(X) \subset \mathrm{Supp}(y_i)$. So assume without loss of generality that $\varphi(X)$ is contained in $y_1 = \ldots = y_s = 0$ only, for some $s \in \{0, \dots, n\}$. Fix $\Phi^* y_i = 0$ for $i \in \{1, \dots, s\}$.

Assume $\Phi^* y_i$ is fixed for all $i \in \{1, \dots, k-1\}$ for some $k \in \{s+1, \dots, n\}$. Let $\mathbb{F} \subset \mathbb{C}(y_{s+1}, \dots, y_n)$ be the subfield generated by degree 0 functions in $\mathbb{C}(y_{s+1}, \dots, y_n)$ and by $y_{s+1}, \dots, y_{k-1}$.

If $y_k \in \mathbb{F}$, then there is a unique way to express $\Phi^* y_k$: write $y_k = \mu \cdot \nu$, where $\mu \in \mathbb{C}(y_{s+1}, \dots, y_n)$ is a monomial of degree 0 and ν is a monomial in $y_{s+1}, \dots, y_{k-1}$. Then $\Phi^* \mu = \varphi^* \mu$ and $\Phi^* \nu$ is already fixed. So set

$$\Phi^* y_k = \varphi^* \mu \cdot \Phi^* \nu.$$

Similarly, if ${y_k}^r \in \mathbb{F}$ for some $r > 0$, then assume r is minimal such r and again write ${y_k}^r = \mu \cdot \nu$, where $\mu \in \mathbb{C}(y_{s+1}, \dots, y_n)$ is a monomial of degree 0 and ν is a monomial in $y_{s+1}, \dots, y_{k-1}$. Then set

$$\Phi^* y_i = \sqrt[r]{\varphi^* \mu \cdot \Phi^* \nu}.$$

Otherwise, if ${y_k}^r \notin \mathbb{F}$ for any $r > 0$, then we have complete freedom to choose $\Phi^* y_k$ to be any homogeneous multi-valued section we like. For instance, we may fix $\Phi^* y_k = 1$.

Proceeding by induction, we eventually fix all $\Phi^* y_1, \dots, \Phi^* y_n$ and hence define the multi-valued map $\Phi\colon \mathbb{C}^m \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\rightrightarrows \mathbb{C}^n$.

We must now show that Φ defined above indeed describes φ. Firstly, we observe Φ satisfies homogeneity condition A3: Let $\mu \in \mathbb{C}(y_{s+1}, \dots, y_n)$ be a monomial of degree 0. Assume there is a nontrivial contribution of y_k in μ and there is no contribution of y_i for $i > k$. Then

$${y_k}^r = \mu \cdot \nu$$

where ν is a monomial in $y_{s+1}, \dots, y_{k-1}$. By our construction:

$$(\Phi^* y_k)^r = \varphi^* \mu \cdot \Phi^* \nu.$$

Therefore

$$\Phi^* \mu = \frac{(\Phi^* y_k)^r}{\Phi^* \nu} = \varphi^* \mu. \tag{4.14}$$

In particular $\Phi^* \mu$ is homogeneous of degree 0, so homogeneity condition A3 holds.

Also the locus $y_1 = \ldots = y_s = 0$ is the (non-empty) toric stratum containing $\varphi(X)$, so Φ satisfies the relevance condition of Definition 4.9.

Finally, by Equation (4.14) the two ring homomorphisms Φ^* and φ^* agree so by Theorem 4.11 indeed Φ describes φ.

□

The descriptions obtained by following the algorithm of this proof are not the favoured ones we discussed in the introduction. For instance for $\varphi = \mathrm{id}_{\mathbb{P}^1}$ we get

$$\begin{aligned} \Phi\colon \mathbb{C}^2 &\to \mathbb{C}^2 \\ [x_1, x_2] &\mapsto [1, \frac{x_2}{x_1}], \end{aligned}$$

and for the embedding $\varphi\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}(1,1,2)$ of §1.1.1, we get

$$\begin{aligned} \Phi\colon \mathbb{C}^2 &\to \mathbb{C}^3 \\ [x_1, x_2] &\mapsto [1, 0, \frac{x_2}{{x_1}^2}]. \end{aligned}$$

We explain how to modify the descriptions obtained here in §4.5.

4.4 The agreement locus revisited

In this section we calculate the agreement locus for any description.

Proposition 4.15. *Let Φ be a description of φ. Then*

$$\mathrm{Agr}(\Phi, \varphi) = \mathrm{Reg}\,\Phi \setminus \Big(\mathrm{Irrel}(X) \;\cup\; \Phi^{-1}\big(\mathrm{Irrel}(Y)\big)\Big).$$

Proof. By the definition of the agreement locus, if $\xi \in \mathrm{Agr}(\Phi, \varphi)$, then

$$\xi \in \mathrm{Reg}\,\Phi \setminus \mathrm{Irrel}(X).$$

The homogeneity condition holds for Φ, so, for such ξ, $\Phi(\xi)$ is contained in a single orbit by condition A4 of Proposition 4.5. Since $\pi_Y(\Phi(\xi))$ is defined it follows that no point in $\Phi(\xi)$ is in $\mathrm{Irrel}(Y)$, which proves the first inclusion:

$$\mathrm{Agr}(\Phi, \varphi) \subset \mathrm{Reg}\,\Phi \setminus \Big(\mathrm{Irrel}(X) \;\cup\; \Phi^{-1}\big(\mathrm{Irrel}(Y)\big)\Big).$$

To prove the other inclusion, take $\xi \in \mathrm{Reg}\,\Phi \setminus \Big(\mathrm{Irrel}(X) \;\cup\; \Phi^{-1}\big(\mathrm{Irrel}(Y)\big)\Big)$ and set $y = \pi_Y(\Phi(\xi)) \in Y$. We must prove, that $x = \pi_X(\xi) \in \mathrm{Reg}\,\varphi$ and that $\varphi(x) = y$, in other words that φ^* maps the local ring $\mathcal{O}_{Y,y} \subset \mathbb{C}(Y)$ into the local ring $\mathcal{O}_{X,x} \subset \mathbb{C}(X)$. So take any $q \in \mathcal{O}_{Y,y}$. By the proof of Theorem 4.11,

$$\varphi^* q = \Phi^* q \quad \text{as elements of } \mathbb{C}(X).$$

Since a lift of y to $\mathbb{C}^m$ is in the image of Φ, it follows that $\Phi^* q$ is defined and hence $\varphi^* q$ is defined. Hence we can calculate:

$$(\varphi^* q)(x) = (\Phi^* q)(\xi) = q(\Phi(\xi)) = q(y),$$

where the outer equalities hold because rational functions can be evaluated on any representative of a point in the Cox cover. Since q is regular at y, also $\varphi^* q \in \mathcal{O}_{X,x}$ as claimed. So $\varphi(x) = y$ and thus $\xi \in \mathrm{Agr}(\Phi, \varphi)$.

□

Corollary 4.16. *The agreement locus* $\operatorname{Agr}(\Phi, \varphi)$ *is an open* G_X*-invariant subset of* $\mathbb{C}^m$ *(and of* $\operatorname{Reg}\Phi$*). In addition, if* X *is* $\mathbb{Q}$*-factorial, then* $\pi_X(\operatorname{Agr}(\Phi, \varphi))$ *is open. In general,* $\pi_X(\operatorname{Agr}(\Phi, \varphi))$ *contains an open dense subset of* X*.*

Proof. $\operatorname{Reg}\Phi$ is an open G_X-invariant subset by Proposition 3.11. $\operatorname{Irrel}(X)$ is clearly closed and G_X-invariant. Finally, $\operatorname{Irrel}(Y)$ is a G_Y-invariant subset of $\mathbb{C}^n$, so by homogeneity condition A4 also $\Phi^{-1}\big(\operatorname{Irrel}(Y)\big)$ is G_X-invariant, and it is closed in $\operatorname{Reg}\Phi$ by Proposition 3.17. Thus $\operatorname{Agr}(\Phi, \varphi)$ is open and G_X invariant by Proposition 4.15.

□

The definition of the agreement locus gives

$$\operatorname{Agr}(\Phi, \varphi) \subset {\pi_X}^{-1}(\operatorname{Reg}\varphi).$$

In §4.5, we distinguish those descriptions for which the equality holds. In the meantime, we call the difference between the two sets the **disagreement locus**. The following statement is essential to prove that complete descriptions exist.

Proposition 4.17. *Let* $\varphi\colon X \dashrightarrow Y$ *be a rational map between two toric varieties* X *and* Y *with a description* $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$*. Consider two open subsets* $U_2 \subset U_1$ *of* $\mathbb{C}^m$*:*

$$U_1 = {\pi_X}^{-1}(\operatorname{Reg}\varphi) \quad \textit{and} \quad U_2 = \operatorname{Agr}(\Phi, \varphi).$$

Then the disagreement locus $D = U_1 \setminus U_2$ *is either empty or is a closed subset in* U_1 *purely of codimension* 1 *in* U_1*.*

Proof. Since U_2 is a non-empty open subset of U_1 by Proposition 4.15 (it is an intersection of three open subsets), clearly D is a proper closed subset in U_1.

By Proposition 4.15:

$$U_2 = \operatorname{Reg}\Phi \setminus \Big(\operatorname{Irrel}(X) \ \cup \ \Phi^{-1}\big(\operatorname{Irrel}(Y)\big)\Big).$$

Note that $\operatorname{Irrel}(X)$ is disjoint from U_1 (because π_X is not regular on $\operatorname{Irrel}(X)$). Therefore

$$D = \underbrace{\Big(U_1 \setminus \operatorname{Reg}\Phi\Big)}_{=:D_{\text{ind}}} \cup \underbrace{\Big(U_1 \cap \Phi^{-1}\big(\operatorname{Irrel}(Y)\big)\Big)}_{=:D_{\text{irrel}}}.$$

By Proposition 3.11 the locus D_{ind} is indeed purely of codimension 1 (or empty). It therefore remains to prove that also D_{irrel} is purely of codimension 1 or empty.

Assume D_{irrel} is not empty and choose arbitrary $\xi \in D_{\text{irrel}}$. We have to prove the codimension of D_{irrel} at ξ is 1. Since $\xi \in U_1$ the rational map φ is regular at

$x = [\xi]$. Consider $y = \varphi(x)$ and its toric open affine neighbourhood $V \subset Y$, such that V is given by non-vanishing of certain coordinates, say

$$V = \{y_1 \neq 0, \dots, y_k \neq 0\} = \{y_1 \cdots y_k \neq 0\}.$$

Set $\gamma = \Phi^*(y_1 \cdots y_k)$. By homogeneity condition A1, there exists $f \in \mathbb{C}[\operatorname{Reg}\Phi]$ such that $\gamma^r = f$ for some $r \geq 1$. We claim that $f(\xi) = 0$ and that for all ξ' in the locus $\{f = 0\}$ and in some sufficiently small open neighbourhood of ξ we have $\xi' \in D_{\mathrm{irrel}}$.

First we prove $f(\xi) = 0$. Since $\xi \in \Phi^{-1}\big(\operatorname{Irrel}(Y)\big)$ it follows $\Phi(\xi)$ and $\operatorname{Irrel}(Y)$ have non-empty intersection. As usual, since $\Phi(\xi)$ is contained in a single torus orbit by the homogeneity condition A4, we have $\Phi(\xi) \subset \operatorname{Irrel}(Y)$. In particular, $\Phi(\xi)$ is disjoint from ${\pi_Y}^{-1}(V)$, in other words the product $y_1 \cdots y_k$ vanishes on $\Phi(\xi)$. So γ vanishes at ξ and therefore f vanishes on ξ.

We prove further that $\xi' \in \{f = 0\}$ implies $\xi' \in D_{\mathrm{irrel}}$, at least on some neighbourhood of ξ. More precisely, we take this neighbourhood to be

$$(\varphi \circ \pi_X)^{-1}(V) \cap \operatorname{Reg}\Phi.$$

Since Φ is regular at such ξ':

$$0 = f(\xi') = \gamma^r(\xi') = \Phi^*(y_1 \cdots y_k)^r(\xi') = (y_1 \cdots y_k)^r(\Phi(\xi')),$$

so $\Phi(\xi')$ is contained in the locus $y_1 \cdots y_k = 0$. Therefore $\Phi(\xi')$ is disjoint from ${\pi_Y}^{-1}(V)$ and hence $\pi_Y(\Phi(\xi'))$ (if non-empty) is not in V. On the other hand $\varphi(x')$ is contained in V by our choice of open neighbourhood of ξ. We conclude, that ξ' cannot be in the agreement locus U_2. But $\xi' \in \operatorname{Reg}\Phi$ and $\xi' \notin \operatorname{Irrel}(X)$ (again by our choice of open neighbourhood of ξ). Therefore by Proposition 4.15 there is no other possibility than $\xi' \in \Phi^{-1}\big(\operatorname{Irrel}(Y)\big)$ so that $\xi' \in D_{\mathrm{irrel}}$ as claimed.

Hence D_{irrel} locally near ξ contains a subset $\{f = 0\}$ purely of codimension 1. Since the same holds true for every $\xi \in D_{\mathrm{irrel}}$ and $D_{\mathrm{irrel}} \neq U_1$, we conclude that D_{irrel} is purely of codimension 1.

□

4.5 Complete descriptions

The map $\Phi(x_1, x_2) = (x_1^3, x_1^2 x_2)$ is a description of the identity map on $\mathbb{P}^1$. As written, it does not evaluate automatically at the point $(0, 1) \in \mathbb{P}^1$: that point is not in the agreement locus. We can modify the description to increase the agreement locus following the usual argument that rational maps are defined in (regular) codimension 1. The divisor (x_1) contains the bad locus, and the components of Φ have multiplicities $\nu_0 = (3, 2)$ along this divisor. Using the exponent vector $\nu' = (1, 0)$ of x_1 itself to push ν_0 down into the span of the

gradings on the Cox ring, $\nu = \nu_0 - \nu' = (3,2) - (1,0) = (2,2)$ computes a scaling factor with which to modify Φ: define

$$\Phi_{\text{new}} = x_1^{-\nu} \cdot \Phi = [x_1, x_2].$$

The agreement locus of Φ_{new} is now as large as it could be, and this new description is better behaved at the point $(0,1)$.

We use this notion of 'complete agreement' to define complete descriptions, and then systematise the argument above to show that complete descriptions exists. In Section 5 we prove a series of additional properties that complete descriptions have.

Definition 4.18. A description Φ of $\varphi\colon X \dashrightarrow Y$ is **complete** if it has the complete agreement property, namely:

C. $\operatorname{Agr}(\Phi, \varphi) = \pi_X^{-1}(\operatorname{Reg}\varphi)$.

Corollary 4.19. *If Φ is a complete description of φ, then*

$$\operatorname{Reg}\varphi = \pi_X(\operatorname{Reg}\Phi \setminus \Phi^{-1}(\operatorname{Irrel}(Y))).$$

In particular φ is regular on X if and only if Φ is regular on $\mathbb{C}^m \setminus \operatorname{Irrel}(X)$ and $\Phi^{-1}(\operatorname{Irrel}(Y))$ is contained in $\operatorname{Irrel}(X)$.

If X is not a product with $\mathbb{C}^*$ as one of factors, then saying Φ is regular on $\mathbb{C}^m \setminus \operatorname{Irrel}(X)$ is equivalent to saying Φ is regular on $\mathbb{C}^m$ (because $\operatorname{Irrel}(X)$ is of codimension at least 2), in which case the regularity criterion for φ is the natural statement one would expect, analogous to standard statement for maps between projective spaces.

Proof. This follows immediately from Proposition 4.15 and the definition of complete description.

□

The crucial claim of this article is that complete descriptions always exist and that they have the properties listed in §1.1. We establish the properties later in Section 5. First we prove the existence.

Let Φ be a description of a rational map of toric varieties $\varphi\colon X \dashrightarrow Y$. If Y is a projective space and Φ is single-valued, then the procedure for computing a complete description of Φ is well known: first clear the denominators in the sequence $\Phi^* y_1, \dots, \Phi^* y_n$ and then divide through by the GCD of the resulting polynomials. The proof of our existence theorem imitates this.

Theorem 4.20. *Let $\varphi\colon X \dashrightarrow Y$ be a rational map of toric varieties. Then there exists a complete description $\Phi\colon \mathbb{C}^m \Lleftarrow \mathbb{C}^n$ of φ.*

Before we start the proof, we discuss the freedom that we have in choosing a description of a rational map. Let Φ be a description of a rational map of toric varieties $\varphi\colon X \dashrightarrow Y$. If $f \in S[X]$ and $w = (w_1, \ldots, w_n)$ is a rational linear combination of $\mathbb{C}^*$-weights of Y, then we can define a multi-valued map

$$\begin{aligned} f^w \cdot \Phi\colon \mathbb{C}^m &\rightrightarrows \mathbb{C}^n \\ x &\mapsto (f^{w_1}\Phi^* y_1, \ldots, f^{w_n}\Phi^* y_n) \end{aligned}$$

which describes the same map φ (this follows easily from the proof of Theorem 4.11). Of course, if $\Phi^* y_i = 0$ for some i, then there is no harm in replacing the i-th coordinate of w with an arbitrary rational number.

More precisely, we consider the n-tuple w as an element of $R_Y \otimes \mathbb{Q} \simeq \mathbb{Q}^n$. We define a map of vector spaces L, whose kernel describes the freedom of taking w. Since, by Proposition 4.10, Φ satisfies the relevance condition B3 of Definition 4.9, there is a smallest cone $\sigma \in \Sigma_Y$ which contains all the rays whose corresponding Cox generators y_i lie in $\ker \Phi^*$. By Corollary 4.12, we may assume that

$$\Phi^* y_i = 0 \iff \rho_i \in \sigma,$$

modifying Φ if necessary.

Let $\operatorname{Star}(\sigma)$ be the star of σ, namely the subfan of Σ_Y comprising those cones that contain σ and their faces. This fan corresponds to the smallest invariant open neighbourhood of the toric stratum containing $\varphi(X)$. Let $\Sigma_{Y(\sigma)}$ be the quotient fan of $\operatorname{Star}(\sigma)$ by σ; this is the fan of the toric stratum containing $\varphi(X)$ regarded as a toric variety in its own right. (If $\varphi(X)$ is not contained in any toric stratum of Y, then both $\operatorname{Star}(\sigma)$ and $\Sigma_{Y(\sigma)}$ are equal to Σ_Y.) Let L be the natural map from $R_Y \otimes \mathbb{Q}$ to the ambient rational vector space of $\Sigma_{Y(\sigma)}$ (the composition of the ray lattice map $R_Y \to N_Y$ and the quotient map). This fits into a diagram of lattices as follows.

$$\begin{tikzcd}
R_Y \arrow[r] \arrow[dr, "L"'] & N_Y \arrow[d] & \text{dual to} & M_Y \\
& \overline{N_{Y(\sigma)}} & \text{dual to} & \overline{M_{Y(\sigma)}} \arrow[u, hook]
\end{tikzcd}$$

where $\overline{N_{Y(\sigma)}}$ is the lattice containing the quotient fan $\Sigma_{Y(\sigma)}$.

Lemma 4.21. *For any $w \in \ker L$ and nonzero $f \in S[X]$, both $f^w \cdot \Phi$ and Φ describe the same map $\varphi\colon X \dashrightarrow Y$. Moreover, the agreement locus of the two descriptions is equal away from the locus $\{f = 0\}$: that is,*

$$\operatorname{Agr}(\Phi, \varphi) \setminus \{f = 0\} = \operatorname{Agr}(f^w \cdot \Phi, \varphi) \setminus \{f = 0\}.$$

Proof. That the two multi-valued maps describe the same map φ follows from the above considerations: the kernel of the ray lattice map gives the freedom to

choose a linear combination of $\mathbb{C}^*$-weights, whereas the pullback of the kernel of the quotient map reflects the freedom to multiply 0 coordinates in the description Φ by anything.

By Proposition 4.15,

$$\begin{aligned} \operatorname{Agr}(\Phi,\varphi) &= \operatorname{Reg}\Phi \setminus \Big(\operatorname{Irrel}(X)\ \cup\ \Phi^{-1}\big(\operatorname{Irrel}(Y)\big)\Big) \quad \text{and} \\ \operatorname{Agr}(f^w\cdot\Phi,\varphi) &= \operatorname{Reg}\,(f^w\cdot\Phi) \setminus \Big(\operatorname{Irrel}(X)\ \cup\ (f^w\cdot\Phi)^{-1}\big(\operatorname{Irrel}(Y)\big)\Big). \end{aligned}$$

Clearly $\operatorname{Reg}\Phi$ and $\operatorname{Reg}\,(f^w\cdot\Phi)$ are equal away from $\{f=0\}$, and also $\operatorname{Irrel}(X)$ does not depend on Φ. Therefore it remains to compare $\Phi^{-1}\big(\operatorname{Irrel}(Y)\big)$ with $(f^w\cdot\Phi)^{-1}\big(\operatorname{Irrel}(Y)\big)$.

Let A be an irreducible component of $\operatorname{Irrel}(Y)$ defined by the vanishing of some coordinates, without loss of generality the coordinates $y_1,\dots,y_s$. Now, for $\xi\in\operatorname{Reg}\Phi$,

$$\xi\in\Phi^{-1}(A)\ \text{ if and only if }\ \Phi^*y_1(\xi)=\cdots=\Phi^*y_s(\xi)=0$$

whereas, for $\xi\in\operatorname{Reg}(f^w\cdot\Phi)$,

$$\xi\in(f^w\cdot\Phi)^{-1}(A)\ \text{ if and only if }\ (f^{w_1}\Phi^*y_1)(\xi)=\cdots=(f^{w_s}\Phi^*y_s)(\xi)=0.$$

Therefore $\Phi^{-1}(A)$ and $(f^w\cdot\Phi)^{-1}(A)$ are equal away from $\{f=0\}$, as claimed. □

Now we are ready to prove the theorem.

Proof of Theorem 4.20. By Theorem 4.13 there is a description

$$\Phi\colon\mathbb{C}^m \dashrightarrow \mathbb{C}^n$$

of φ. By Proposition 4.17, the disagreement locus

$$D=\pi_X^{-1}(\operatorname{Reg}\varphi)\setminus\operatorname{Agr}(\Phi,\varphi)$$

is a union of codimension 1 components. If D is empty, then the theorem is proved, so suppose it is not empty; we must modify Φ so that the new description is defined on those components which cover the locus where φ is defined.

Choose any homogeneously prime component of D and pick a homogeneously irreducible polynomial $f\in S[X]$ that vanishes along it. We aim to replace Φ by $f^w\cdot\Phi$ for some vector w so that $\operatorname{Agr}(f^w\cdot\Phi,\varphi)$ contains a general point of $\{f=0\}$.

Step 1: interpret disagreement in terms of a fan. Let $v_i\in\mathbb{Q}$ be the multiplicity of f in Φ^*y_i and consider $v=(v_1,\dots,v_n)$ as a point in $R_Y\otimes\mathbb{Q}$, where R_Y is the ray lattice of Y.

Lemma 4.22. *Let m be a form on the lattice containing $\Sigma_{Y(\sigma)}$, and χ^m be the corresponding rational function on Y. Then the order of vanishing of $\varphi^*\chi^m$ along the divisor (f) is equal to $\langle L(v), m\rangle$. In particular, $L(v)$ is an integral point in the lattice of $\Sigma_{Y(\sigma)}$.*

Proof. L^*m is the monomial expressed in terms of Cox coordinates of Y. So

$$\varphi^*\chi^m = \Phi^*\chi^{L^*m}.$$

Now the order of $\Phi^*y_i = \Phi^*\chi^{e_i}$ along (f) is by definition $v_i = \langle v, e_i\rangle$, so the order of $\Phi^*\chi^{L^*m}$ along (f) is

$$\langle v, L^*m\rangle = \langle L(v), m\rangle .$$

□

Corollary 4.23. *If $L(v)$ is not in the support of $\Sigma_{Y(\sigma)}$, then φ is not regular on (f).*

Proof. Let τ be any cone in $\Sigma_{Y(\sigma)}$. Since $L(v) \notin \tau$, there exists $m_\tau \in \tau^\vee$ such that $\langle L(v), m_\tau\rangle < 0$. Then by Lemma 4.22 the rational function $\varphi^*\chi^{m_\tau}$ has a pole along (f). Let U_τ be the affine open subset corresponding to a cone in $\mathrm{Star}(\tau)$, which maps to τ. Note that the collection of such U_τ for all $\tau \in \Sigma_{Y(\sigma)}$ will cover the image of φ. By Proposition 2.18, this implies that φ is not regular on (f).

□

Thus if $L(v)$ does not lie in the support of $\Sigma_{Y(\sigma)}$, then (f) is not part of the disagreement locus, contradicting our initial setup. In short, we may assume that $L(v)$ lies the support of $\Sigma_{Y(\sigma)}$.

Step 2: modify Φ. Let τ_{quo} be the cone in $\Sigma_{Y(\sigma)}$ of minimal dimension that contains $L(v)$, and τ_{star} be a cone in $\mathrm{Star}(\sigma)$ that maps exactly onto τ_{quo} and is maximal with this property.

By definition of τ_{star}, there is a vector $u \in \tau_{star}$ that maps to $L(v)$, and so by choosing a vector v' of $R_Y \otimes \mathbb{Q}$ in the hyperplane quadrant above τ_{star} which maps to u, we have $v - v' \in \ker L$. We may assume that the coordinates of this vector $v' = (v'_1, \ldots, v'_n)$ satisfy $v'_i = 0$ if the i^{th} ray of Σ_Y is not in τ_{star} and otherwise $v'_i \geq 0$. We define

$$\Phi_{\text{new}} := f^{v'-v} \cdot \Phi.$$

By Lemma 4.21 the two descriptions of φ have the same (dis)agreement locus away from $\{f = 0\}$.

Step 3: $\mathrm{Agr}(\Phi_{\mathrm{new}}, \varphi)$ **contains a general point of** $\{f = 0\}$**.** By Proposition 4.15, it is enough to prove the following two statements:

- Φ_{new} is regular on a general point of (f).
- Φ_{new} does not map general point of (f) into the irrelevant locus of Y.

The first is immediate: $f^{-v} \cdot \Phi$ is regular along (f), since f does not appear in any component $\Phi^* y_i$, and as each component v'_i of v' is non-negative, Φ_{new} is also regular there. Moreover, this shows that if $x \in \{f = 0\}$ is a general point, then $\Phi_{\mathrm{new}}(x)$ has zero y_i-coordinate if and only if either $\Phi^* y_i = 0$ or $v'_i > 0$. In particular, if the i-th ray of Σ_Y is not in τ_{star}, then $\Phi_{\mathrm{new}}(x)$ has non-zero i-th coordinate. This means that the standard generator of B_Y determined by τ_{star} is nonzero at $\Phi_{\mathrm{new}}(x)$, and so $\Phi_{\mathrm{new}}(x)$ is not in the irrelevant locus of Y. Therefore $\mathrm{Agr}(\Phi_{\mathrm{new}}, \varphi)$ contains a general point of $\{f = 0\}$ as claimed.

Thus we have obtained a description Φ_{new} of φ whose disagreement locus contains one component less than that of Φ. Continuing inductively, we obtain a description with an empty disagreement locus, namely a complete description. $\square$

Example 4.24. Complete descriptions are not unique. For example, take X to be $\mathbb{C}$ with coordinate x and Y to be the non-$\mathbb{Q}$-factorial base of the standard flop from §1.1.5: $S[Y] = \mathbb{C}[y_1, y_2, y_3, y_4]$ graded by $\mathbb{Z}$ in degrees $(1, 1, -1, -1)$. Then the map

$$\begin{aligned} X &\longrightarrow Y \\ [x] &\longmapsto [x^t, x^t, x^{1-t}, x^{1-t}] \end{aligned}$$

is a complete description for any rational t in the interval $[0, 1]$.

If the target is $\mathbb{Q}$-factorial, and the map is regular in codimension 1, then a complete description is unique up to multiplication by scalars using the whole group action, but we do not use this fact.

5 Geometry of descriptions

In this section, we prove that images and preimages of subschemes behave as well as the first examples could allow, and we compute descriptions of compositions of maps, where composition makes sense. We work throughout with a rational map φ together with a description Φ (not necessarily a complete description, unless explicitly mentioned) as in the diagram

$$\begin{tikzcd} \mathbb{C}^m \arrow[r, "\Phi"] \arrow[d, dashed, "\pi_X"] & \mathbb{C}^n \arrow[d, dashed, "\pi_Y"] \\ X \arrow[r, dashed, "\varphi"] & Y \end{tikzcd}$$

In §1.1 we insisted that descriptions should behave well when pulling back Cartier divisors. We prove this 'local Cartier pullback' property now, and then present a few additional conditions below that are closely related to the complete agreement property that characterises complete descriptions. Recall the notion of ceiling and floor from §2.3: if δ is a homogeneous multi-valued section in the field of fractions of $\Gamma(\Phi)$, then $\lfloor\delta\rfloor$ and $\lceil\delta\rceil$ are both homogeneous (single-valued) sections in $S(X)$.

Proposition 5.1. *Let $D = (f)$ be a Weil divisor on Y, for some $f \in S(Y)$, whose support does not contain $\varphi(\operatorname{Reg}\varphi)$. Consider an open subset $V \subset Y$ for which $D|_V$ is Cartier. Denote the interior of $\pi_X(\operatorname{Agr}(\Phi, \varphi))$ by $\mathfrak{agr} \subset X$ and let $U = \varphi^{-1}(V) \cap \mathfrak{agr}$. Write $\Phi^* f = \lceil\Phi^* f\rceil \cdot \gamma$ for some homogeneous multi-valued section γ on X.*

Then γ is invertible on ${\pi_X}^{-1}(U)$ and the Cartier divisor ${\varphi|_U}^*(D|_V)$ on U is equal to the restriction $E|_U$, where $E = (\lceil\Phi^* f\rceil)$ denotes the divisor on X defined by $\lceil\Phi^* f\rceil$.

Note that if Φ is a complete description, then $\mathfrak{agr} = \operatorname{Reg}\varphi$ and so $U = \varphi^{-1}(V)$. Also if D is a Cartier divisor on Y, then we may take $V = Y$. Thus, if both of these hold, the statement of the proposition has a much easier form; see condition D below. We observe the following lemma before we prove the proposition.

Lemma 5.2. *Let δ be a homogeneous multi-valued section in the field of fractions of $\Gamma(\Phi)$. If $W \subset \mathbb{C}^m$ is an open subset on which δ is invertible, then $\lfloor\delta\rfloor, \lceil\delta\rceil \in S[X]$ are also invertible on W.*

Proof. By definition, $\delta = \sqrt[r]{g}$ is invertible on W if and only if $g \in S(X)$ is invertible on W. For some (reduced) $f \in S[X]$ the locus $Z = \{f = 0\}$ is the codimension 1 locus of $\mathbb{C}^m \setminus W$, so that $\mathcal{O}_{\mathbb{C}^m}(W) = S[X][f^{-1}]$. Now g is invertible on W if and only if $g, g^{-1} \in S[X][f^{-1}]$. By Proposition 2.27, $S[X][f^{-1}] \subset \Gamma(\Phi)[f^{-1}]$ is a simple ring extension, so $\delta, \delta^{-1} \in \Gamma(\Phi)[f^{-1}]$ by Definition 2.24(iii). Thus by Proposition 2.28 both $\lfloor\delta\rfloor$ and $\lceil\delta\rceil$ are invertible elements in $S[X][f^{-1}] = \mathcal{O}_{\mathbb{C}^m}(W)$, and so they are both invertible on W as claimed.

□

Proof of Proposition 5.1. We first work locally on an open subset $V' \subset V$ where $D|_{V'}$ is principal and defined by $h \in \mathbb{C}(Y)$. Set $k = h/f$. By construction, $k \in S(Y)$ is invertible on ${\pi_Y}^{-1}(V')$. Suppose $U' = \varphi^{-1}(V') \cap \mathfrak{agr}$. We claim that $\Phi^* k$ is invertible on $W' := {\pi_X}^{-1}(U')$. To show this, we simply check that $(\Phi^* k)(\xi)$ is nonzero for any $\xi \in W'$. But $(\Phi^* k)(\xi) = k(\eta)$ for any $\eta \in \Phi(\xi)$, and for such η we have $\pi_Y(\eta) = \varphi \circ \pi_X(\xi) \in V'$ so $k(\eta) \neq 0$. Thus $\Phi^* k$ is invertible. It follows from Lemma 5.2 that $\lfloor\Phi^* k\rfloor$ is also invertible on W'.

Since $f = h/k$ and $\Phi^* h = \varphi^* h$,

$$\Phi^* f = \frac{\varphi^* h}{\Phi^* k} \qquad \text{and} \qquad \lceil\Phi^* f\rceil = \frac{\varphi^* h}{\lfloor\Phi^* k\rfloor}.$$

It then follows from $\Phi^* f = \lceil \Phi^* f \rceil \cdot \gamma$ that

$$\gamma = \frac{\lfloor \Phi^* k \rfloor}{\Phi^* k},$$

and so γ is invertible on W' and $\varphi|_{U'}{}^*(D|_{U'}) = E|_{U'}$.

The same conclusion is true for any $V' \subset V$ on which $D|_{V'}$ is principal. Since such V' cover V and the corresponding U' cover U, it follows that γ is invertible on U and $\varphi|_U{}^*(D|_U) = E|_U$ as claimed.

$\square$

Definition 5.3. Let Φ be a description of $\varphi\colon X \dashrightarrow Y$. We recall the complete agreement property C of Definition 4.18 and define some other properties of Φ:

C. **Complete agreement**: $\operatorname{Agr}(\Phi, \varphi) = \pi_X^{-1}(\operatorname{Reg}\varphi)$.

D. **Global Cartier pullback**: Let $D = (f)$ be a Cartier divisor on Y for some $f \in S(Y)$ whose support does not contain $\varphi(\operatorname{Reg}\varphi)$. Write $\Phi^* f = \lceil \Phi^* f \rceil \cdot \gamma$ for a homogeneous multi-valued section γ on X. Let $E = (\lceil \Phi^* f \rceil)$ be the divisor on X defined by $\lceil \Phi^* f \rceil$. Then γ is invertible on $\pi_X{}^{-1}(\operatorname{Reg}\varphi)$ and the Cartier divisor $\varphi^* D$ on $\operatorname{Reg}\varphi$ is equal to the restriction $E|_{\operatorname{Reg}\varphi}$.

E. **Weil preimage:** If $D = (f)$ is an effective Weil divisor on Y for some function $f \in S[Y]$ and $\varphi(\operatorname{Reg}\varphi)$ is not contained in the support of D, then $\Phi^* f$ is regular on $\operatorname{Reg}\varphi$ and its set-theoretic zero locus agrees with the set $\varphi^{-1}(D)$, the preimage of the support of D.

F. **Coordinate divisors preimage:** If $D = (y_i)$ is the coordinate divisor for some $i \in \{1, \dots, n\}$ and $\varphi(\operatorname{Reg}\varphi)$ is not contained in the support of D, then $\Phi^* y_i$ is regular on $\operatorname{Reg}\varphi$ and its set-theoretic zero locus agrees with the set $\varphi^{-1}(D)$, the preimage of the support of D.

Condition F is the same as condition E but only for divisors of the form $D = (y_i)$. Note that in Condition D, if X has no torus factors and φ is regular (or at least regular in codimension 1), then γ is necessarily a constant in $\mathbb{C}$.

Proposition 5.4. *Let Φ be a description of $\varphi\colon X \dashrightarrow Y$. We have the following implications between the properties of Definition 5.3:*

$$\mathrm{E} \Longrightarrow \mathrm{F} \Longrightarrow \mathrm{C} \Longrightarrow \mathrm{D}.$$

If, furthermore, Y is $\mathbb{Q}$-factorial, then D $\Longrightarrow$ E, *so that all conditions* C, D, E, F *are equivalent.*

Proof. The implication E $\Rightarrow$ F is clear.

Assume F holds so that $\Phi^* y_i$ is regular on $\pi_X^{-1}(\operatorname{Reg}\varphi)$. So, in particular,

$$\operatorname{Reg}\Phi \supset \pi_X^{-1}(\operatorname{Reg}\varphi).$$

Now assume (by changing the order of coordinates, if necessary) that $y_1, \ldots, y_s$ define a component of the irrelevant locus of Y. Then the intersection

$$(y_1) \cap \cdots \cap (y_s)$$

of divisors on Y is empty, and so the intersection

$$\varphi^{-1}((y_1)) \cap \cdots \cap \varphi^{-1}((y_s))$$

is also empty as a subset of $\operatorname{Reg}\varphi$. So the zero locus of $\Phi^* y_1, \ldots, \Phi^* y_s$ does not intersect $\pi_X^{-1}(\operatorname{Reg}\varphi)$. Proposition 4.15 now implies that $\operatorname{Agr}(\Phi, \varphi) = \pi_X^{-1}(\operatorname{Reg}\varphi)$, and so property C holds. The implication C $\Rightarrow$ D follows from Proposition 5.1, with $V = Y$ and $\mathfrak{agr} = \operatorname{Reg}\varphi$.

Finally assume Y is $\mathbb{Q}$-factorial and property D holds. Then, since every Weil divisor is $\mathbb{Q}$-Cartier, property E follows automatically.

$\square$

5.1 Composition of descriptions

We define the composition $\Psi \circ \Phi$ of multi-valued maps $\Phi \colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ and $\Psi \colon \mathbb{C}^n \rightrightarrows \mathbb{C}^p$. Recall the notation $\Gamma(\Phi)$ from §3.3 for the map ring of a multi-valued map Φ.

Definition 5.5. Let $\Phi \colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ and $\Psi \colon \mathbb{C}^n \rightrightarrows \mathbb{C}^p$ be two multi-valued which both satisfy the homogeneity condition and for which $\Phi^{-1}(\operatorname{Reg}\Psi)$ is non-empty. Define a map $(\Psi \circ \Phi)^* \colon S[X] \to \overline{S(X)}$ by

$$(\Psi \circ \Phi)^* := \widetilde{\Phi^*} \circ \Psi^*,$$

where $\widetilde{\Phi^*}$ is an extension of $\Phi^* \colon S[Y] \to \Gamma(\Phi)$ to $\Gamma(\Psi)$ as in Lemma 2.33:

$$\begin{tikzcd}
S[Z] \arrow[dr, "\Psi^*"] & S[Y] \arrow[d, hook] \arrow[dr, "\Phi^*"] & S[X] \arrow[d, hook] \\
& \Gamma(\Psi) \arrow[dr, dotted, "\widetilde{\Phi^*}"] & \Gamma(\Phi) \arrow[d, hook] \\
& & \overline{S(X)}
\end{tikzcd}$$

This map $(\Psi \circ \Phi)^*$ defines a simple multi-valued map $\Psi \circ \Phi \colon \mathbb{C}^m \rightrightarrows \mathbb{C}^p$, since the homogeneity condition A1 implies that pullbacks $(\Psi \circ \Phi)^*(z_i)$ of coordinate functions z_i are homogeneous multi-valued sections.

Proposition 5.6. *In the setup of Definition 5.5, the composite map $\Psi \circ \Phi$ satisfies the homogeneity condition. Moreover, for any set $A \subset \mathbb{C}^m$,*

$$(\Psi \circ \Phi)(A) = \Psi(\Phi(A)).$$

In particular, the image of any subset is independent of the choice of the extension $\widetilde{\Phi^*}$ of Φ^* that was made.

Proof. $\Psi \circ \Phi$ satisfies the homogeneity condition A1 because Ψ^* and $\widetilde{\Phi^*}$ both pull back homogeneous elements to homogeneous elements, and $(\Psi \circ \Phi)^*$ is their composition. The image condition also follows immediately from the construction and Proposition 3.18—we could use the same map ring for both Φ and $\Psi \circ \Phi$.

□

Theorem 5.7. *Let X, Y and Z be toric varieties, $\varphi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow Z$ be two rational maps such that the composition $\psi \circ \varphi$ is defined, and Φ and Ψ be descriptions of φ and ψ respectively. If either*

- $\Phi(\operatorname{Reg}\Phi) \cap \operatorname{Reg}\Psi$ *is non-empty or*
- Ψ *is a complete description,*

then $\Psi \circ \Phi$ describes $\psi \circ \varphi$.

Proof. If Ψ is a complete description, then $\Phi(\operatorname{Reg}\Phi)$ meets $\pi_Y^{-1}(\operatorname{Reg}\psi) = \operatorname{Agr}(\Psi, \psi)$, so the first condition also holds. So in either case we can at least define the composition $\Psi \circ \Phi$.

Since in general $\operatorname{Agr}(\Psi, \psi)$ is an open subset in $\pi_Y^{-1}(\operatorname{Reg}\psi)$, the intersection of $\Phi(\operatorname{Reg}\Phi)$ and $\operatorname{Agr}(\Psi, \psi)$ is nonempty and therefore, $\Psi \circ \Phi$ is relevant. It is also homogeneous by Proposition 5.6. Hence by Theorem 4.11, it is a description of the rational map corresponding to the ring homomorphism $(\Psi \circ \Phi)^*\colon R \to K(X)$, where R is a subring of $K(Z)$. Since $(\Psi \circ \Phi)^*$ was constructed as the composition of $\Phi^* \circ \Psi^*$ and since Φ and Ψ describe φ and ψ respectively, it follows that

$$(\Psi \circ \Phi)^* = \varphi^* \circ \psi^* = (\psi \circ \varphi)^*$$

on the subring of rational functions where each of the maps make sense. Hence $\Psi \circ \Phi$ describes $\psi \circ \varphi$ as claimed.

□

Here we observe that in certain circumstances a composition of two complete descriptions of maps is a complete description of the composition.

Lemma 5.8. *Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be two rational maps such that the composition $\psi \circ \varphi$ makes sense. Assume*

$$\operatorname{Reg}(\psi \circ \varphi) = \varphi^{-1}(\operatorname{Reg}\psi)$$

(note that $\supset$ always holds, so in fact we only assume $\subset$ in the above equality). Furthermore assume Φ and Ψ are complete descriptions of φ and ψ respectively. Then $\Psi \circ \Phi$ is a complete description of $\psi \circ \varphi$.

We skip the proof, as it is a standard verification of the definitions.

□

5.2 Image of a subscheme

Suppose $A \subset X$ is a closed subscheme defined by a homogeneous ideal $I_A \lhd S[X]$. We consider the problem of finding the ideal in $S[Y]$ of the scheme-theoretic image of A under $\varphi\colon X \dashrightarrow Y$. Recall from §2.1.3 the notation $\overline{\varphi}|_U(A)$ for the closure of the image of $\varphi|_U(A \cap U)$ where $U \subset \operatorname{Reg}\varphi$ is open.

We define an ideal $J_A \lhd S[Y]$ by

$$J_A := \Big((\Phi^*)^{-1}\left(\langle I_A\rangle_{\Gamma(\Phi)}\right)\Big)^{\mathrm{hgs}},$$

the homogeneous preimage of the ideal that I_A generates in the map ring.

Theorem 5.9. *Let $\varphi\colon X \dashrightarrow Y$ be a rational map of toric varieties with a description $\Phi\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$, and let $\mathfrak{agr}$ be the interior of $\pi_X(\operatorname{Agr}(\Phi,\varphi))$; in particular, $\mathfrak{agr} \subset \operatorname{Reg}\varphi$.*

Suppose $A \subset X$ is a closed subscheme defined by a homogeneous, saturated ideal $I_A \lhd S[X]$, and define J_A as above. Then the scheme-theoretic image $\overline{\varphi}|_{\mathfrak{agr}}(A) \subset Y$ and the subscheme $B \subset Y$ defined by J_A are equal. In particular

(i) *B is independent of the choice of map ring $\Gamma(\Phi)$ (and of the choice of the saturated ideal I_A defining A).*

(ii) *If Φ is a complete description of φ, then $\overline{\varphi}(A)$ and B are equal.*

Proof. Let V be a standard open affine toric subset of Y given by nonvanishing of some coordinates, say

$$V = \{y \in Y \mid y_i \neq 0 \text{ for every } i \in E\}$$

where $E \subset \{1,\dots,n\}$ is some appropriate subset. Denoting

$$\upsilon = \prod_{i\in E} {y_i}^{r_i},$$

where r_i are the minimal positive integers such that $\Phi^* {y_i}^{r_i} \in \mathbb{C}[\operatorname{Reg}\Phi]$, we set $\mathcal{O}_Y(V)$ to be the homogeneous localisation of $S[Y]$ at υ so that $V = \operatorname{Spec}\mathcal{O}_Y(V)$. Of course, such open subsets form an open cover of Y. It is enough to prove that $B \cap V = \overline{\varphi}|_{\mathfrak{agr}}(A) \cap V$, which we do below by comparing their ideals in $\mathcal{O}_Y(V)$.

In the first place, suppose $\Phi^*\upsilon = 0$. Then by Corollary 4.12 the locus $\overline{\varphi}(X)$ is disjoint from V, so in this case we need to prove $B \cap V = \emptyset$. But $\upsilon \in (\Phi^*)^{-1}\left(\langle I_A\rangle_{\Gamma(\Phi)}\right)$, so $\upsilon \in J_A$, and indeed $B \cap V = \emptyset$.

So assume that $\Phi^*\upsilon \neq 0$ and consider $\varphi^{-1}(V) \cap \mathfrak{agr}$. It is an open subset of X, and thus it has a covering by open affine subsets of X. Thus by Lemma 2.11 there exists a finite subset $G \subset S[X]$, such that $\varphi^{-1}(V) \cap \mathfrak{agr} = \bigcup_{g\in G} X_g$ and each $X_g = X \setminus \operatorname{Supp}(g)$ is affine. Then $X_g = \operatorname{Spec}\mathcal{O}_X(X_g)$ where $\mathcal{O}_X(X_g)$ is the homogeneous localisation of $S[X]$ at g.

For any $g \in G$, the following diagram shows the natural relationships between subrings of a common field $\overline{S(X)}$ on the left and subrings of $S(Y)$ on the right.

$$\begin{tikzcd}
\mathcal{O}_X(X_g) \arrow[d, hook, "\deg 0 \text{ part}"] & & \mathcal{O}_Y(V) \arrow[ll, "\varphi^*"'] \arrow[d, hook, "\deg 0 \text{ part}"'] \\
S[X][g^{-1}] & & S[Y][\upsilon^{-1}] \\
\mathbb{C}[\operatorname{Reg}\Phi] \arrow[u, hook] \arrow[d, hook] & & S[Y] \arrow[u, hook] \arrow[dll, "\Phi^*"'] \\
\Gamma(\Phi) & &
\end{tikzcd} \tag{5.10}$$

Since $\Phi^*(\upsilon) \neq 0$, it is natural to extend the domain of Φ^* to $S[Y][\upsilon^{-1}]$ (we don't need to specify the precise subset of $\overline{S(X)}$ that is the image of these elements). With that, by Theorem 4.11, we have

$$\Phi^* f = \varphi^* f \quad \text{for all } f \in \mathcal{O}_Y(V).$$

In this sense Diagram (5.10) is commutative. Moreover, since $\varphi(X_g) \subset V$ and ${\pi_X}^{-1}(X_g) \subset \operatorname{Agr}(\Phi,\varphi)$, it follows that $\Phi^*(\upsilon)$ is invertible in $S[X][g^{-1}]$.

It is enough to prove that the following two ideals in $\mathcal{O}_Y(V)$ are equal:

$$I(\overline{\varphi}|_{\mathfrak{agr}}(A) \cap V) = \bigcap_{g\in G} (\varphi^*)^{-1}\left((I_A)_{(g)}\right)$$
$$\text{and} \quad I(B) = (J_A)_{(\upsilon)}.$$

The intersection consists of precisely those functions on V whose preimage in any X_g is in the ideal of A there—which is why it defines the image.

We redraw Diagram (5.10) marking where each ideal lives:

$$
\begin{tikzcd}
(I_A)_{(g)} \lhd & \mathcal{O}_X(X_g) \arrow[d, hook, "\deg 0\ \text{part}"] & & \mathcal{O}_Y(V) \arrow[ll, "\varphi^*"'] \arrow[d, hook, "\deg 0\ \text{part}"'] & \rhd\, I(\overline{\varphi}|_{\mathfrak{agr}}(A) \cap V), I(B) \\
\langle I_A \rangle_{S[X][g^{-1}]} \lhd & S[X][g^{-1}] & & S[Y][\upsilon^{-1}] & \rhd\, \langle J_A \rangle_{S[Y][\upsilon^{-1}]} \\
I_A \lhd & \mathbb{C}[\operatorname{Reg}\Phi] \arrow[u, hook] \arrow[d, hook] & & S[Y] \arrow[u, hook] \arrow[dll, "\Phi^*"'] & \rhd\, J_A \\
\langle I_A \rangle_{\Gamma(\Phi)} \lhd & \Gamma(\Phi) & & &
\end{tikzcd}
$$

The idea of the proof is now straightforward: grab an element q in one of the ideals $I(\overline{\varphi}(A))$ or $I(B)$ and drag it around Diagram (5.10) to see that in fact q is also in the other ideal. We exploit the "commutativity" of the diagram and our choice that $\Phi^*(\upsilon)$ is a homogeneous single-valued section which is invertible on X_g. Here are the details.

Take $q \in \mathcal{O}_Y(V)$. Then $q \in I(B)$ if and only if $q = \tilde{q}/\upsilon^l$ for some $\tilde{q} \in J_A$ and $l \in \mathbb{Z}$, so:

$$
\begin{aligned}
q \in I(B) &\iff \Phi^*(q \cdot \upsilon^l) \in \langle I_A \rangle_{\Gamma(\Phi)} \\
&\iff \varphi^*(q) \cdot \Phi^*(\upsilon^l) \in \langle I_A \rangle_{\Gamma(\Phi)}.
\end{aligned}
$$

Since $\Phi^*(\upsilon^l) \in \mathbb{C}[\operatorname{Reg}\Phi]$ by Corollary 2.25, we have $\varphi^*(q) \cdot \Phi^*(\upsilon^l) \in \mathbb{C}[\operatorname{Reg}\Phi]$. At this point, our insistence that $\Gamma(\Phi)$ is a simple extension is key. By Corollary 2.31 we can continue the chain of equivalences:

$$
\ldots \iff \varphi^*(q) \cdot \Phi^*(\upsilon^l) \in I_A.
$$

But $\Phi^*(\upsilon^l)$ is invertible on each X_g, so we continue:

$$
\ldots \iff \varphi^*(q) \in \langle I_A \rangle_{S[X][g^{-1}]} \qquad \text{for every } g \in G.
$$

The implication $\Longleftarrow$ above needs a careful explanation, as this implication does not hold if I_A is not saturated (as in Example 2.16, say). We postpone the proof of this implication until later, meanwhile we continue the series of implications:

$$
\begin{aligned}
\ldots &\iff \varphi^*(q) \in (I_A)_{(g)} && \text{for every } g \in G \\
&\iff q \in (\varphi^*)^{-1}\left((I_A)_{(g)}\right) && \text{for every } g \in G \\
&\iff q \in I(\overline{\varphi}|_{\mathfrak{agr}}(A \cap V)).
\end{aligned}
$$

It remains to prove the missing implication for $q = \tilde{q}/\upsilon^l$ as above:

$$\varphi^*(q) \in \langle I_A \rangle_{S[X][g^{-1}]} \text{ for every } g \in G \Longrightarrow \varphi^*(q) \cdot \Phi^*(\upsilon^l) \in I_A.$$

Let $\hat{A} \subset \operatorname{Reg}\Phi$ be the subscheme defined by $\langle I_A \rangle_{\mathbb{C}[\operatorname{Reg}\Phi]}$. Suppose $U_g = \{g \neq 0\} \subset \mathbb{C}^m$. The claim of the implication is that if $\varphi^*(q)$ vanishes on $\hat{A} \cap U_g$ for all $g \in G$, then it vanishes on $\hat{A} \cap (\operatorname{Reg}\Phi \cap \{\Phi^*\upsilon \neq 0\})$. Since I_A is saturated, $\hat{A} = \overline{\hat{A} \setminus \operatorname{Irrel}(X)}$, where the closure is taken in $\operatorname{Reg}\Phi$, so it is enough to prove the following inclusion of open subsets:

$$(\operatorname{Reg}\Phi \setminus \operatorname{Irrel}(X)) \cap \{\Phi^*\upsilon \neq 0\} \subset \bigcup_{g \in G} U_g.$$

Suppose $\xi \in (\operatorname{Reg}\Phi \setminus \operatorname{Irrel}(X)) \cap \{\Phi^*\upsilon \neq 0\}$. Then $\Phi^*\upsilon(\xi) \neq 0$ and $\upsilon(\Phi(\xi)) \neq 0$, so $\pi_Y \circ \Phi(\xi) \in V$. In particular $\Phi(\xi)$ is not contained in $\operatorname{Irrel}(Y)$ and by Proposition 4.15, $\xi \in \operatorname{Agr}(\Phi, \varphi)$ and $\pi_Y \circ \Phi(\xi) = \varphi_{\mathrm{reg}} \circ \pi_X(\xi)$. Thus $\pi_X(\xi) \in \varphi_{\mathrm{reg}}^{-1}(V)$ and there exists $g \in G$ such that $\pi_X(\xi) \in X_g$, so in particular $g(\xi) \neq 0$ and thus $\xi \in U_g$, as claimed.

□

5.3 Preimage of a subscheme

Consider as usual a rational map of toric varieties $\varphi \colon X \dashrightarrow Y$ with a description $\Phi \colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ and fixed choice of map ring $\Phi^* \colon S[Y] \longrightarrow \Gamma(\Phi)$. We study the problem of finding the preimage of a closed subscheme $B \subset Y$ under φ. Our main interest is to compute $\varphi_{\mathrm{reg}}^{-1}(B)$, the scheme-theoretic preimage under $\varphi_{\mathrm{reg}} \colon \operatorname{Reg}\varphi \to Y$, but inevitably the subschemes of X we define are concerned with the closure of this.

5.3.1 The regular preimage ideal J_B

Suppose that B is defined by the ideal $I_B \lhd S[Y]$. We consider a related ideal $J_B \lhd \mathbb{C}[\operatorname{Reg}\Phi]$ which is the intersection of the ideal in $\Gamma(\Phi)$ generated by $\Phi^*(I_B)$ with $\mathbb{C}[\operatorname{Reg}\Phi]$:

$$J_B = \mathbb{C}[\operatorname{Reg}\Phi] \cap \langle \Phi^*(I_B) \rangle_{\Gamma(\Phi)} \lhd \mathbb{C}[\operatorname{Reg}\Phi].$$

We refer to J_B as **the regular preimage ideal**.

One might expect to need an elimination calculation, using Gröbner basis say, to compute the generators of J_B, since its definition involves the intersection of an ideal and a ring. But it can be avoided, just as it can when considering analogous ideals coming from maps to ordinary projective spaces. We check this first: the calculation of J_B depends only on Φ being homogeneous. Recall from §2.3 the notion of ceiling $\lceil \gamma \rceil$ of a multi-valued section γ.

Proposition 5.11. *Let $I_B = \langle f_1, \ldots, f_\beta \rangle \lhd S[Y]$ be a homogeneous ideal generated by homogeneous sections f_i. Suppose that $\Phi\colon \mathbb{C}^m \multimap \mathbb{C}^n$ is a multi-valued map satisfying a homogeneity condition* A *(this would hold, for instance, if Φ were the description of some rational map $\varphi\colon X \dashrightarrow Y$). Then*

$$J_B = \left\langle \lceil \Phi^* f_1 \rceil, \ldots, \lceil \Phi^* f_\beta \rceil \right\rangle \quad \textit{as an ideal of } \mathbb{C}[\operatorname{Reg}\Phi].$$

It follows that the assignment $I_B \to J_B$ is additive: that is, for homogeneous ideals $I_{B_1}, I_{B_2} \subset S[Y]$,

$$\mathbb{C}[\operatorname{Reg}\Phi] \cap \langle \Phi^*(I_{B_1} + I_{B_2}) \rangle_{\Gamma(\Phi)} = J_{B_1} + J_{B_2}.$$

Proof. Follows immediately from the definitions and Corollary 2.31.

□

Note also that J_B does not depend on the choice of map ring $\Gamma(\Phi)$ (see Proposition 2.30).

5.3.2 Computable preimages

The relationship between the preimage $\overline{\varphi_{\mathrm{reg}}^{-1}(B)}$ and the regular preimage ideal J_B is a little delicate—we have already seen a counter-example to an optimistic statement in §1.1.4—and so we identify a general property which will permit computation of preimages under certain conditions. The dependence on the defining ideal I_B is important.

Definition 5.12. Let $\varphi\colon X \dashrightarrow Y$ be a rational map of toric varieties with a description $\Phi\colon \mathbb{C}^m \multimap \mathbb{C}^n$. Fix a closed subscheme $B \subset Y$ with homogeneous defining ideal $I_B \lhd S[Y]$. Let J_B be the regular preimage ideal as defined above.

For any open subset $W \subset Y$, we say B **has a computable preimage on W with respect to I_B** (and with respect to Φ and $\Gamma(\Phi)$) if and only if the subscheme of X defined by J_B equals $\varphi_{\mathrm{reg}}^{-1}(B)$ on $\varphi_{\mathrm{reg}}^{-1}(W)$.

The particular description Φ we are working with at any time is fixed, so we do not usually mention Φ. As stated, this property also depends on the choice of map ring $\Gamma(\Phi)$, but this is illusory and we also do not mention it; see Corollary 5.14 below.

Theorem 5.13. *Let $\varphi\colon X \dashrightarrow Y$ be a rational map of toric varieties with a description $\Phi\colon \mathbb{C}^m \multimap \mathbb{C}^n$. Let $B \subset Y$ be a closed subscheme with homogeneous defining ideal $I_B = \langle f_1, \ldots, f_\beta \rangle \subset S[Y]$. If $W \subset Y$ is an open subset on which each divisor $(f_i)|_W$ is Cartier and $\pi_X^{-1}\varphi_{\mathrm{reg}}^{-1}(W) \subset \operatorname{Agr}(\Phi, \varphi)$, then B has a computable preimage on W.*

Moreover, on the interior of $\pi_X(\operatorname{Agr}(\Phi, \varphi))$ the scheme defined by J_B is a subscheme of $\varphi_{\mathrm{reg}}^{-1}(B)$.

The main content of this result is that our ability to compute a preimage for B depends in part on the equations we use to define B.

Corollary 5.14. *Let X, Y, φ, Φ and B be as in the theorem.*

(i) *If Φ is a complete description, then the subscheme B has computable preimage on the smooth locus Y_0 of Y.*

(ii) *If Φ is a complete description and I_B freely defines B, then B has a computable preimage on Y.*

(iii) *If Φ is a complete description and $\varphi_{\text{reg}}^{-1}(B) = \overline{\varphi_{\text{reg}}^{-1}(B \cap Y_0)}$ (which happens for instance, when B is disjoint from the singularities of Y), then B has a computable preimage on Y*

The conditions do not impose requirements on the existence of many Cartier divisors on Y (although Cartier divisors are required by part (ii) implicitly).

The proof of the main part of the theorem is in two steps which we state as separate lemmas. The first step reduces the theorem to the case where I_B is a principal ideal. In the second we observe that the computable preimage property holds on the Cartier locus of principal ideals. The proof of the "moreover" part of the theorem very similar to the proof of Theorem 5.9, so we omit it: one would choose a suitable cover of open affine subsets of X and Y, and prove the appropriate inclusion of ideals; the calculations may be simplified by proving additivity (in the same way as Lemma 5.15 below) and reducing to the case where I_B is a principal ideal.

Lemma 5.15. *Having a computable preimage is additive in the following sense. Let B_1 and B_2 be two subschemes in Y. Suppose $W \subset Y$ is an open subset on which both B_1 and B_2 have a computable preimage with respect to their defining ideals $I_{B_1}, I_{B_2} \subset S[Y]$ respectively. Then the closed subscheme $B_1 \cap B_2$ has a computable preimage on W with respect to $I_{B_1} + I_{B_2}$.*

Proof. Let $B = B_1 \cap B_2 \subset Y$. It is enough to prove that $J_B = J_{B_1} + J_{B_2}$, since this sum defines the intersection of the preimages of B_1 and B_2 on the open subset $\varphi_{\text{reg}}^{-1}(W)$ (see Lemma 2.13). The equality $J_B = J_{B_1} + J_{B_2}$ is observed in Proposition 5.11.

$\square$

Lemma 5.16. *If $f \in S[Y]$ is a polynomial and $W \subset Y$ an open subset on which the restriction $(f)|_W$ of the Weil divisor (f) on Y is Cartier and $\pi_X^{-1}\varphi_{\text{reg}}^{-1}(W) \subset \text{Agr}(\Phi, \varphi)$, then B has computable preimage on W with respect to its defining ideal $I_B = \langle f \rangle$.*

Proof. By Proposition 5.11, J_B is principal and generated by $\lceil \Phi^* f \rceil$. By Proposition 5.1 we have $\Phi^* f = \lceil \Phi^* f \rceil \cdot \gamma$, where γ is a homogeneous multi-valued section invertible on $\pi_X^{-1} \varphi_{\mathrm{reg}}^{-1}(W)$. Moreover $\lceil \Phi^* f \rceil$ defines the divisor $\varphi^* D$ on $\varphi^{-1}(W)$. Since the definition of pullback of a Cartier divisor agrees with the definition of preimage of the underlying scheme, it follows that $\varphi^{-1}(B)$ is given by the ideal J_B on W.

□

6 Computability and examples

6.1 Cox coordinates and computability

The main theorem gives an effective algorithm for computing complete descriptions of rational maps. Given a description, our results show how to determine the regularity locus of the underlying map and how to calculate image and preimage of subschemes in an algorithmic way.

We have not made a serious study of the complexity of these algorithms, but it is easy to get an informal sense of where the computation lies. The calculation of a map ring $\Gamma(\Phi)$ and its generators γ_i is the only computationally expensive calculation that arises in this general framework for describing maps between toric varieties that is not already needed when describing maps between ordinary projective spaces or products of them. It is clear from the constructive proofs of existence that this only requires GCD calculations, so it shouldn't be regarded as formidable. We have implemented this in the computational algebra system Magma [BCP97], and in practical examples it seems that factorising polynomials in place of some GCDs is effective, since the number of GCD calculations blows up with the number of Cox coordinates—at first sight we need all pairwise GCDs.

6.2 Further examples

We do several blow ups and give their complete description.

6.2.1 Three toric blow ups

These are well-known examples where the use of radical expressions is already entrenched. In our language, these radical weighted blow ups are complete descriptions.

Standard blow up of $\mathbb{C}^3$ at the origin Let X be the toric variety with Cox ring $\mathbb{C}[x_1, x_2, x_3, x_4]$, irrelevant ideal $B_X = (x_1, x_2, x_3)$ and grading $(1, 1, 1, -1)$. The variety X is the standard blow up of $\mathbb{C}^3$ at the origin, and it has complete

description:

$$\begin{aligned} X &\longrightarrow \mathbb{C}^3 \\ [x_1, x_2, x_3, x_4] &\longmapsto [x_1x_4, x_2x_4, x_3x_4]. \end{aligned}$$

Blow up of $\frac{1}{2}(1,1,1)$ singularity Let Y be the standard affine $\frac{1}{2}(1,1,1)$ singularity: $S[Y] = \mathbb{C}[y_1, y_2, y_3]$ with trivial irrelevant ideal and grading $\frac{1}{2}(1,1,1)$. The standard weighted blow up of the origin has domain X with Cox ring $S[X] = \mathbb{C}[x_1, x_2, x_3, x_4]$, irrelevant ideal $B_X = (x_1, x_2, x_3)$ and grading $(1,1,1,-2)$. The blow up has complete description:

$$\begin{aligned} X &\longrightarrow Y \\ [x_1, x_2, x_3, x_4] &\longmapsto [x_1{x_4}^{\frac{1}{2}}, x_2{x_4}^{\frac{1}{2}}, x_3{x_4}^{\frac{1}{2}}]. \end{aligned}$$

Weighted blow ups of $\frac{1}{4}(1,1,2)$ singularity Let Y be the affine $\frac{1}{4}(1,1,2)$ singularity: $S[Y] = \mathbb{C}[y_1, y_2, y_3]$ with trivial irrelevant ideal and grading $\frac{1}{4}(1,1,2)$. The weighted blow up $bl_{(1,1,2)}$ of the origin has complete description

$$\begin{aligned} X &\longrightarrow Y \\ [x_1, x_2, x_3, x_4] &\longmapsto [x_1{x_4}^{\frac{1}{4}}, x_2{x_4}^{\frac{1}{4}}, x_3{x_4}^{\frac{1}{2}}]. \end{aligned}$$

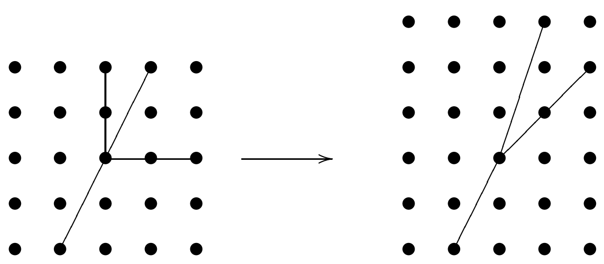
where X has $S[X] = \mathbb{C}[x_1, x_2, x_3, x_4]$, irrelevant ideal $B_X = (x_1, x_2, x_3)$ and grading $(1,1,2,-4)$.

6.2.2 Reading toric birational maps with complete descriptions

Let $X = \mathbb{P}(1,1,2)$ with Cox coordinates x_1, x_2, x_3 and Y be the toric variety with Cox coordinates y_1, y_2, y_3, y_4, bi-grading given by the matrix of weights

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and irrelevant ideal} \quad B_Y = (y_1, y_2) \cap (y_3, y_4).$$

Suppose X and Y are described by fans in a common lattice $N = \mathbb{Z}^2$ as follows.



The implicit birational map between X and Y is

$$\begin{aligned} \varphi\colon X &\dashrightarrow Y \\ [x_1, x_2, x_3] &\longmapsto [x_1, x_1x_2, x_1x_3, x_1x_2] \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi\colon Y &\dashrightarrow X \\ [y_1, y_2, y_3, y_4] &\longmapsto [{y_1}^2y_4, y_2y_4, y_1y_2y_3y_4], \end{aligned}$$

for indeed the compositions are

$$[x_1, x_2, x_3] \longmapsto [x_1, x_1x_2, x_1x_3, x_1x_2] \longmapsto [{x_1}^3x_2, {x_1}^2{x_2}^2, {x_1}^4{x_2}^2x_3] = [x_1, x_2, x_3]$$

and

$$\begin{aligned}[y_1, y_2, y_3, y_4] \longmapsto [{y_1}^2y_4, y_2y_4, y_1y_2y_3y_4] \longmapsto & [{y_1}^2y_4, {y_1}^2y_2{y_3}^2, {y_1}^3y_2y_3{y_4}^2, {y_1}^2y_2{y_4}^2] \\ &= [y_1, y_2, {y_1}^3y_2y_3{y_4}^2, {y_1}^3y_2{y_4}^3] \\ &= [y_1, y_2, y_3, y_4],\end{aligned}$$

using the two rows of the gradings of Y to simplify the final expression in two steps.

The easiest way to confirm that the descriptions satisfy the homogeneity and relevance conditions is to compare them to descriptions of the same maps by rational functions: φ can be described by $\Phi = (1, x_2/x_1, x_3/{x_1}^2, x_2/x_1)$ or by $\Phi = (1, x_2/x_1, 1, x_1x_2/x_3)$, which looks nicer but has smaller regular locus $\operatorname{Reg}\Phi$ (but still a codimension 1 complement); ψ can be described by $\Psi = (1, y_2/{y_1}^2, y_2y_3/({y_1}^3y_4))$.

The geometry of this birational equivalence evident in the fans but it is not clear from these descriptions. It is better seen using complete descriptions. We can make complete descriptions easily from the original monomial descriptions (which are both regular on the respective entire GIT covers). The question is simply how much can we cancel. For φ we can use the first grading of Y to remove a $\sqrt{x_1}$ factor, and then the second grading to remove a further x_1: thus

$$\begin{aligned}[x_1, x_1x_2, x_1x_3, x_1x_2] \text{ becomes } & [\sqrt{x_1}, x_2, x_1x_3, x_1x_2\sqrt{x_1}] \\ & \text{which in turn becomes } [\sqrt{x_1}, x_2, x_3, x_2\sqrt{x_1}].\end{aligned}$$

Similarly we can modify the description of ψ, so the result is

$$\begin{aligned}\varphi\colon [x_1, x_2, x_3] &\longmapsto [\sqrt{x_1}, x_2, x_3, x_2\sqrt{x_1}] \\ \psi\colon [y_1, y_2, y_3, y_4] &\longmapsto [y_1^2\sqrt{y_4}, y_2\sqrt{y_4}, y_1y_2y_3].\end{aligned}$$

Many features of the birational geometry are now clear. The map φ is not defined at the three 0-strata of X, while ψ is not defined on the loci $\operatorname{Supp}(y_1) \cap \operatorname{Supp}(y_4)$ and $(y_2) \cap (y_4)$ in Y. The coordinate loci (x_1) and (x_2) in X are contracted, and similarly (y_1), (y_2) and (y_4) in Y are contracted. Furthermore, comparing with the weighted blow ups above, we see that (x_1) and (y_4) are contracted as $\frac{1}{2}(1,1)$ exceptional divisors, while (y_1) is a $(2,1)$ weighted blow up of a smooth point, and (y_2) and (x_2) are ordinary (smooth) blow ups of smooth points.

6.2.3 Toric del Pezzo surface in a weighted projective 5-space

Fix two toric varieties X and Y and a map $\alpha\colon \operatorname{Pic} Y \to \operatorname{Pic} X$. We can use the structure of descriptions to classify all the regular maps $\varphi\colon X \to Y$ for which

$\varphi^* = \alpha$. We present an easy example to illustrate the technique. For brevity, we will assume that the image of φ is not contained in any toric stratum of Y, not even after a change of coordinates on Y.

Let $X = \mathbb{F}_1$, simply $\mathbb{P}^2$ blown up in a single point, a del Pezzo surface of degree 8. Thus

$$S[X] = \mathbb{C}[x_1, x_2, x_3, x_4] \quad \text{graded by } \mathbb{Z}^2 \text{ with gradings} \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and irrelevant ideal $(x_1, x_2) \cap (x_3, x_4)$. Consider also $Y = \mathbb{P}(1, 1, 1, 2, 2, 2)$:

$$S[Y] = \mathbb{C}[y_1, y_2, y_3, y_4, y_5, y_6] \quad \text{graded by } \mathbb{Z} \text{ with gradings} \quad \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

For the demonstration, we assume that the regular map $\varphi \colon X \to Y$ pulls back a divisor in $\mathcal{O}_Y(2)$ (the ample generator of $\operatorname{Pic} Y$) to an anticanonical divisor of X in $\mathcal{O}(3, 2)$. We claim that in such case the map φ is unique up to changes of coordinates on X and Y. (This is analogous to nondegenerate quadratic maps $\mathbb{P}^1 \to \mathbb{P}^2$ boiling down to the usual conic: the nominated linear systems in source and target have the same dimension in each case, so the only concern for us is to handle the 'de-Veronese' process on Y.)

So let $\Phi \colon \mathbb{C}^4 \multimap \mathbb{C}^6$ be a complete description of φ. By condition D the multivalued sections $(\Phi^* y_1)^2$, $(\Phi^* y_2)^2$, $(\Phi^* y_3)^2$, $\Phi^* y_4$, $\Phi^* y_5$ and $\Phi^* y_6$ are all single valued. Applying the homogeneity condition A2 we see that $\frac{\Phi^* y_2}{\Phi^* y_1}$ and $\frac{\Phi^* y_3}{\Phi^* y_1}$ are also single-valued. Corollary 4.19 implies that all the $\Phi^* y_i$ are regular. Thus we can write

$$\begin{aligned} \Phi \colon \mathbb{C}^m &\multimap \mathbb{C}^n \\ x &\mapsto [f_1\sqrt{g}, f_2\sqrt{g}, f_3\sqrt{g}, f_4, f_5, f_6] \end{aligned}$$

for polynomials $f_i, g \in S[X]$.

Applying the homogeneity condition A2 once more to this general form of the description, we see that

$$\begin{aligned} \deg f_1 &= \deg f_2 = \deg f_3, \\ \deg f_4 &= \deg f_5 = \deg f_6, \\ \text{and} \quad 2 \deg f_1 + \deg g &= \deg f_4 = (3, 2). \end{aligned}$$

The last condition narrows the possibilities for the multidegree of f_1:

$$\deg f_1 \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$

But the linear systems in multidegrees $(0, 0), (1, 0), (0, 1)$ are small, and allowing the degree of f_1 to be any those would force the 3 sections f_1, f_2, f_3 to be linearly dependent. A suitable coordinate change on Y would then transform (at least)

one of f_1, f_2, f_3 to 0, presenting the image of φ inside some toric stratum, which is exactly what our simplifying assumption forbids. So $\deg f_1 = (1,1)$.

So $\deg g = (1,0)$, and changing coordinates on X we may assume $g = x_1$. Also the $\mathbb{C}$-linear span of f_1, f_2, f_3 is equal to the span x_1x_4, x_2x_4, x_3, so changing coordinates on Y we may assume

$$f_1 = x_1x_4, \quad f_2 = x_2x_4 \quad \text{and} \quad f_3 = x_3.$$

The linear system $(3,2)$ is spanned by the nine monomials:

$$x_1{x_3}^2, x_2{x_3}^2, {x_1}^2x_3x_4, x_1x_2x_3x_4, {x_2}^2x_3x_4, {x_1}^3{x_4}^2, {x_1}^2x_2{x_4}^2, x_1{x_2}^2{x_4}^2, {x_2}^3{x_4}^2.$$

However if any of f_4, f_5, f_6 contains any summand divisible by x_1 then we can change the coordinates on Y to get rid of this summand. For instance if $f_4 = x_2{x_3}^2 + x_1x_2x_3x_4$, then $f_4 - (f_2\sqrt{g})(f_3\sqrt{g}) = x_2{x_3}^2$. Therefore we might assume f_4, f_5, f_6 are contained in span of $x_2{x_3}^2, {x_2}^2x_3x_4, {x_2}^3{x_4}^2$ and changing coordinates on Y again we may assume

$$f_4 = x_2{x_3}^2, \quad f_5 = {x_2}^2x_3x_4 \quad \text{and} \quad f_6 = {x_2}^3{x_4}^2.$$

Thus every map φ satisfying the assumptions can be written as

$$\begin{aligned} \varphi\colon X &\longrightarrow Y \\ x &\longmapsto [x_1x_4\sqrt{x_1}, x_2x_4\sqrt{x_1}, x_3\sqrt{x_1}, x_2{x_3}^2, {x_2}^2x_3x_4, {x_2}^3{x_4}^2] \end{aligned}$$

in some homogeneous coordinates on X and Y.

6.2.4 A complete toric variety with no Cartier divisors

Consider the fan in $\mathbb{Z}^3$ spanned by a cube with vertices $(\pm1, \pm1, \pm1)$ but with one of the rays perturbed; for example $(1,-1,-1)$ replaced by $(2,-1,-1)$. This gives a fan F_Y generated by 6 non-simplicial 3-dimensional cones which corresponds to a complete, non-projective toric variety Y with $\operatorname{Pic} Y \cong 0$. This can be checked by hand, but we support ourselves with the toric geometry package in the Magma computer algebra system [BCP97].

We build the fan F_Y by nominating rays and cones.

```
> rays := [ [ 1,1,1], [ 1,1,-1], [ 1,-1,1], [ 2,-1,-1],
>           [-1,1,1], [-1,1,-1], [-1,-1,1], [-1,-1,-1] ];
> cones := [ [1,2,3,4],[5,6,7,8],[1,2,5,6],
>            [3,4,7,8],[1,3,5,7],[2,4,6,8] ];
> FY := Fan( rays, cones );
```

The fan determines the toric variety Y (that we define over $\mathbb{Q}$ and assign names to the Cox coordinates), and we compute some of its properties.

```
> Y<y1,y2,y3,y4,y5,y6,y7,y8> := ToricVariety( Rationals(), FY );
> IsComplete(Y);
true
> IsProjective(Y);
false
> PicardLattice(Y);
0-dimensional toric lattice Pic0
```

We describe a 1-dimensional family of regular maps $\varphi_a \colon X \to Y$ where $X = \mathbb{P}^1 \times \mathbb{P}^1$. A general member of this family is birational, but a special member is generically 4 to 1. Suppose $\mathbb{A}^1$ is an affine line with coordinate a and describe $\varphi \colon X \times \mathbb{A}^1 \to Y$ by

$$\varphi \colon (x_1, x_2, x_3, x_4, a) \longmapsto (1,\ x_1,\ 1,\ (x_3 + a x_4)\sqrt[3]{x_3},\ a{x_3}^2 + {x_4}^2,\ 1,\ x_2,\ \sqrt[3]{{x_3}^2}).$$

Construct the product $XA = X \times \mathbb{A}^1$ with coordinate a on the last factor (we use $*$ to denote product of toric varieties).

```
> P1 := WeightedProjectiveSpace( Rationals(), [1,1] );
> A1 := ToricVariety( Rationals(), FanOfAffineSpace(1) );
> XA<x1,x2,x3,x4, a> := P1 * P1 * A1;
```

We define Φ as above: the radical expressions themselves are self-explanatory, and the initial `FamilyOfMultivaluedSections(XA)` declaration is computational hygiene that asserts that everything in the sequence should be interpreted as a multi-valued section.

```
> Phi := [ FamilyOfMultivaluedSections(XA) |
>               1, x1,  1, MultivaluedSection(x3,1/3)*(x3+a*x4),
>   a*x3^2 + x4^2,  1, x2, MultivaluedSection(x3,2/3)];
> phi := ToricVarietyMap( XA, Y, Phi );
> phi;
A map between toric varieties described by:
    1,
    (x1),
    1,
    (x3)^(1/3)*(x3 + x4*a),
    (x3^2*a + x4^2),
    1,
    (x2),
    (x3)^(2/3)
```

(We note that there are 3 values of a, such that the map φ_a is not regular.) First consider the special map $\varphi_0 \colon X \to Y$ obtained by substituting $a = 0$:

$$\varphi_0 \colon (x_1, x_2, x_3, x_4) \longmapsto (1,\ x_1,\ 1,\ {x_3}^{4/3},\ {x_4}^2,\ 1,\ x_2,\ {x_3}^{2/3}).$$

The image of this map is contained in the irreducible divisor (f), where $f = {y_1}^2{y_3}^2y_4 - {y_6}^2{y_8}^2$.

```
> X<x1,x2,x3,x4> := P1*P1;
> i0 := ToricVarietyMap( X, XA, [x1,x2,x3,x4,0] );
> phi0 := i0 * phi;
> f := y1^2*y3^2*y4 - y6^2*y8^2;
> IsHomogeneous(Y,f);
true
> phi0 * f;
(0)
> IsRegular(phi0);
true
```

Since f is a binomial, the normalisation of the support $\mathrm{Supp}(f)$ is a toric variety. We recover this toric variety Z, together with its embedding ψ into Y.

```
> supp := Scheme(Y, f);
> Z<[x]>, psi := BinomialToricEmbedding(supp);
```

This function in general computes an embedding from a toric variety whose image ideal is contained in the ideal of a given scheme: it interprets the binomial equations defining a scheme as those of a toric embedding, and pulls the remaining equations back to the implicit toric variety. The variety Z is this pullback and ψ is the toric embedding. In this case, the entire image scheme is toric, so Z is the required toric variety, but regarded as being defined by the zero equation in an ambient toric variety. So we simply look at that ambient space.

```
> Ambient(Z);
Toric variety of dimension 2
Variables: z[1], z[2], z[3], z[4]
The components of the irrelevant ideal are:
    (z[3], z[2]), (z[4], z[1])
The quotient grading is:
    1/2( 0, 0, 1, 1 )
The 2 integer gradings are:
         0, 1, 1, 0,
         1, 0, 0, 1
> CompleteDescription(psi);
[
    1,
    (z[4]),
    1,
    (z[3])^(2/3),
    (z[2]),
```

```
    1,
    (z[1]),
    (z[3])^(1/3)
]
```

We observe the similarity between φ_0 and ψ. In fact Z is a quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by a $\mathbb{Z}_2$-action and φ_0 is a composition of a 2-to-1 map $X \to X$, another 2-to-1 map $X \to Z$ and ψ. Since ψ is the normalisation map, it is birational onto its image (in fact one can check that it is an isomorphism onto the image). Thus φ_0 is generically 4-to-1.

Now consider φ_1 (obtained by substituting $a = 1$ to φ).

```
> i1 := ToricVarietyMap( X, XA, [x1,x2,x3,x4,1] );
> phi1 := i1 * phi;
> CompleteDescription(phi1);
[
    1,
    (x1),
    1,
    (x3)^(1/3)*(x3 + x4),
    (x3^2 + x4^2),
    1,
    (x2),
    (x3)^(2/3)
]
> IsRegular(phi1);
true
```

Consider the point $p := [1,1,1,4,10,1,1,1] \in Y$, with $p = \varphi_1([1,1,1,3])$ and consider the ideal I generated by ${y_1}^3 y_2 y_3 y_5 - 10 y_6 y_7 {y_8}^3$, ${y_2}^2 {y_6}^2 - {y_3}^2 {y_7}^2$, $-\frac{1}{4} {y_1}^2 {y_2}^2 y_4 + {y_7}^2 {y_8}^2$. This is a radical, homogeneous ideal, defining a scheme of dimension 0 in Y and containing p (in fact this ideal is a homogeneous ideal defining p with reduced scheme structure).

```
> P:=Scheme(Y, [ y1^3*y2*y3*y5 - 10*y6*y7*y8^3,
>                y2^2*y6^2 - y3^2*y7^2,
>                -1/4*y1^2*y2^2*y4 + y7^2*y8^2 ] );
> Dimension(P);
0
> IsReduced(P);
true
> Y![1,1,1,4,10,1,1,1] in P;
true (1 : 1 : 1 : 4 : 10 : 1 : 1 : 1)
```

Also p is a smooth point of Y. Thus by Corollary 5.13(iii) the ideal J_p defines the preimage scheme ${\varphi_1}^{-1}(p)$.

```
> preimage := P @@ phi1;
> J := Ideal(preimage);
> Saturation( J, IrrelevantIdeal(X) );
Ideal of Polynomial ring of rank 4 over Rational Field
Order: Lexicographical
Variables: x1, x2, x3, x4
Homogeneous
Basis:
[
    x1 - x2,
    x3 - 1/3*x4
]
```

Thus ${\varphi_1}^{-1}(p)$ is the reduced scheme supported on $[1,1,1,3]$. Since the dimension and length of the preimage of point can only jump at special points, it follows that φ_1 is birational onto its image.

6.2.5 Multi-valued multi-linear systems

It is worth noting for a final time that the homogeneity conditions are more precise than simply the arranging for the degrees of components of a map being correct.

Let $X = \mathbb{P}(1,1,2)$ with coordinates x_1, x_2, x_3 and $Y = \mathbb{P}(1,2,3)$ with coordinates y_1, y_2, y_3. Let $f = {x_1}^3 - x_2x_3$ and $\gamma = \sqrt{f}$. Then

$$\Phi\colon (x_1, x_2, x_3) \mapsto (\sqrt{x_1}, x_2, \gamma)$$

has the correct degrees but nevertheless fails to determine a rational map: indeed

$$\Phi^*(y_2/{y_1}^2) = x_2/x_1 \quad \text{is nice, but} \quad \Phi^*(y_3/{y_1}^3) = \sqrt{1 - \frac{x_2x_3}{{x_1}^3}}$$

is not a rational function on X. (Or, using the homogeneity condition A1 instead, $\Phi^*({x_1}^3 + x_3)$ is not a homogeneous multi-valued section.) Simply arranging for the correct homogeneous degrees is not the full content of the homogeneity condition. It is better thought of as requiring all defining sections to be elements of a single vector space of multi-valued sections together with its multiples. If γ is the third coordinate, then the degree 3 sections defining the map must all have γ as their common irrational part, by Proposition 3.6 as usual. But they do not: $\Phi^*(y_1y_2) = \sqrt{x_1} \cdot x_2$ has a different irrational part. Forcing $\Phi^* y_3 = \gamma$ will put $\sqrt[r]{f}$ for $r = 6, 4$ as a factor into the first two components, but then we can scale the entire irrational part away in any case.

However, defining the map as

$$\Phi\colon (x_1, x_2, x_3) \mapsto (\gamma, {x_2}^3, \gamma^3 + \gamma x_1 x_3)$$

is fine, since now

$$\Phi^*(y_2/{y_1}^2) = {x_2}^3/f \quad \text{and} \quad \Phi^*(y_3/{y_1}^3) = 1 + (x_1x_3/f).$$

(And, at least as a first test, $\Phi^*(y_1^3 + y_3)$ is now $\gamma \cdot (2f + x_1x_3)$, which is a homogeneous multi-valued section.)

If we regard a map to a weighted projective space as being determined by a basis of a graded ring

$$V = \bigoplus_{d \in \mathbb{N}} V_d$$

where each $V_d \subset \overline{S(X)}$ is a finite-dimensional vector space consisting only of multi-valued sections of degree d/N, for some fixed denominator $N \in \mathbb{N}$, then we must ensure that each V_d has the same irrational part γ^d, for some $\gamma \in \overline{S[X]}$. In the corrected example, this holds:

$$V_1 = \gamma \cdot \mathbb{C}, \quad V_2 = \gamma^2 \cdot \mathbb{C}\left\langle 1, {x_2}^3/f \right\rangle, \quad V_3 = \gamma^3 \cdot \mathbb{C}\left\langle 1, {x_2}^3/f, x_1x_3/f \right\rangle$$

and so on—the irrational parts of these spaces of sections are visibly the same (up to the power that fixes their degree).

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